2.10 Mathematical arguments – The second tour of the Pigeonhole principle

1. Many characteristics can be assigned to the pigeons and holes; what is appropriate depends on the problem at hand. Some ideas include ordering within holes, forcing distinctness, and operations on pigeons. We present a few more examples.

   Let $S = \{(x, y) \mid x, y \text{ are integers and } -2 \le x, y \le 2\}$ be a set of points in the plane, and let $P_1, P_2, \ldots, P_6$ be any six points in S. Prove that there are three distinct indices i, j, k such that the area of triangle $P_iP_jP_k$ is at most 2. (The area could be 0; that is, these three points are collinear.)

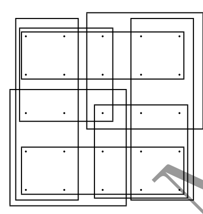
   There are many approaches to this problem. Most of the approaches are based on the following geometric observation: It suffices to show that triangle $P_iP_jP_k$ is inscribed in a rectangle of area at most 4. At this particular moment, we are going to assume this fact and explore several proofs based on this fact and some nice applications of the Pigeonhole principle. (We encourage the interested reader to prove this fact directly.)

   Use the following pair of diagrams to show the result. One might want to start the proof with "We consider two cases. In the first case, we assume that there is at least one chosen point in one of the squares A, B, C, D."

2. (Continuation) Use the following pair of diagrams to show the result. One might start the proof as "We consider two cases. In the first case, we assume that there are no more than two chosen points in each of the rectangles A and B."

3. (Continuation) Use the following pair of diagrams to show the result.

4. (Continuation) Use the following diagram to show the result.



5. (Continuation) Use the following pair of diagrams to show the result.

2.11 Challenges in enumerative counting (part 4)

1. Matt will arrange six identical dominoes on a 6×7 chessboard so that a path is formed from the upper left corner to the lower right corner. Any two consecutive dominoes have an edge in common. How many distinct arrangements are possible?

2. Six computers $C_1, C_2, \ldots, C_6$ form a network. Each computer C_i, $1 \le i \le 6$, is connected with three other computers $C_{i-1}, C_{i+1}, C_{i+3}$ (where $C_{k+6} = C_k$). In how many ways can one upload one of three antivirus softwares (say, McAfee, Webroot, PCtools) to each computer so that connected machines always have different antivirus softwares?

3. Let $ABCD$ be a regular tetrahedron. Each edge of the tetrahedron is colored, independently and randomly, either red or blue. What is the probability that there is a path from A to B consisting of red edges only?

4. Mr. Porter has 12 students in his combinatorics class. In the first week of class, students form four groups of three to work on a reading assignment. In the second week, students form six groups of two to work on a study report. If no pairs of students can be in the same group for both weeks, in how many ways can the students form the groups in the second week?

5. Determine the number of subsets S of $\{1, 2, \ldots, 9999\}$ satisfying the following conditions:

   (a) it has 2015 elements;

   (b) the sum of the elements in any subset of S is not divisible by 2016.

2.18 Brain teasers and mathematical reasoning (part 6)

1. Each side and diagonal of a regular dodecagon is colored in one of 12 possible colors. (All 12 colors need not be used.)

   (a) Suppose that some color, say red, is either the most popular color or one of the most popular colors. That is, there are as many red segments as there are segments of any other given color. What is the least possible number of red segments?

   (b) Suppose that some color, say blue, is either the least popular color or one of the least popular colors. That is, for each color other than blue, there are as many segments of that color as there are blue segments. What is the greatest possible number of blue segments?

   (c) Suppose that at least three colors are used. Is it possible that for any three colors, there exist three vertices which are joined with each other by segments of these three colors?

2. Show that the following list contains an even number of odd numbers.

$$\binom{2015}{1}, \binom{2015}{3}, \binom{2015}{5}, \ldots, \binom{2015}{2013}, \binom{2015}{2015}.$$

3. Find the smallest prime that is the sixth term of an increasing arithmetic sequence, all five preceding terms also being prime.

4. A math contest has 100 participants, and consists in finding the answers to three questions. At the end of the contest, one notices that there are a total of exactly 200 correct answers. Prove that one can find 34 participants who have correct answers to the same two questions (and, possibly, to the third question).

5. Determine if there exists a 10-tuple $(a_1, a_2, \ldots, a_{10})$ of (nor necessary distinct) non-negative integers such that

$$n = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_{10}},$$

   where (a) $n = 1707$ (b) $n = 1776$ (c) $n = 1826$

   (By the way, the three greatest mathematicians Leonhard Euler, Carl Friedrich Gauss, and Bernhard Riemann, were born in 1707, 1776, and 1826, respectively.)

3.2 Mathematical arguments – Coloring and assigning weights (part 2)

1. There are n numbers written on a circle. Each time, one chooses 4 consecutive numbers in clockwise order, say, a, b, c, d, and switches the positions of b and c only if both $b > c$ and $a < d$. Prove the operations will terminate in a finite number of steps.

2. Sam draws an $m \times n$ grid of dots on the coordinate plane, at the points (a, b) where $1 \le a \le m$ and $1 \le b \le n$. He proceeds to draw a cyclic path along k of these dots, (a_1, b_1), (a_2, b_2), $\ldots$, (a_k, b_k), such that (a_i, b_i) and (a_{i+1}, b_{i+1}) are 1 unit apart for each i, and also (a_k, b_k) and (a_1, b_1) are 1 unit apart. Sam makes sure his path does not cross itself, that is, the k dots are distinct. Find, with proof, the maximum possible value of k in terms of m and n.

3. Given are 100 coins in a row in the following order: heads, tails, heads, $\ldots$, tails. We can select several consecutive coins and turn each one of them up side down. Find the minimum number of operations necessary to turn all coins tails up.

4. Two positive integers are written on the board. The following operation is repeated: if $a < b$ are the numbers on the board, then a is erased and $ab/(b-a)$ is written in its place. At some point the numbers on the board are equal. Prove that again they are positive integers.

5. Each cell of a $2n \times 2n$ table is filled with one of the numbers -1 or 1 in such a way that:

   (a) the sum of the numbers in each row and column is equal to 0;

   (b) if the number of entries on the diagonal is even, then the sum of its entries is equal to 0.

   Prove that n must be an even number.

   Note: The rows and columns of a $2n \times 2n$ table are each labelled 1 to $2n$ in a natural order. Thus each cell corresponds to a pair of positive integer (i, j) with $1 \le i, j \le 2n$. For $n > 1$, the table has $8n - 4$ diagonals of two types. A diagonal of first type consists all cells (i, j) for which $i + j$ is a constant, and the diagonal of this second type consists all cells (i, j) for which $i - j$ is constant.