1.13 The first look at the Cauchy-Schwarz inequality (part 2)

1. Show that for positive real numbers a, b, c,

$$\left(\frac{a}{2}\right)^2 + \left(\frac{b}{3}\right)^2 + \left(\frac{c}{6}\right)^2 \ge \left(\frac{a+b+c}{7}\right)^2$$

2. Let a, b, c be positive real numbers. Prove the inequality $a^2 + b^2 + c^2 \ge ab + bc + ca$ first by AM-GM/completing the squares and then by Cauchy-Schwarz.

3. Let p be a polynomial with positive real coefficients. Prove that $p(x^2)p(y^2) \ge (p(xy))^2$ for any positive real numbers x and y.

4. Let $x, y, z > 1$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$. Prove that

$$\sqrt{x+y+z} \ge \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

5. Determine the maximum value of $y = \sqrt{5x} + \sqrt{3-3x}$.

1.14 Algebra practice set 5

1. Fresh Mann said, "The function $f(x) = ax^2 + bx + c$ passes through 6 points. Their x-coordinates are consecutive positive integers, and their y-coordinates are $34, 55, 84, 119, 160$, and 207, respectively." Sophy Moore replied, "You've made an error in your list," and replaced one of Fresh Mann's numbers with the correct y-coordinate. Find the corrected value.

2. Let a, b, c be real numbers. Prove that

$$(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$$

always has at least one real root.

3. Let $a_1, a_2, \ldots, a_{100}$ be a sequence of numbers such that $a_1 = 100$ and $a_1 + a_2 + \cdots + a_n = n^2 a_n$ for $n = 1, 2, \ldots, 100$. Compute a_{100}.

4. Given nonzero real numbers a, b, and c such that the quadratic equations (in x)

$$ax^2+bx+c=0, \quad bx^2+cx+a=0, \quad cx^2+ax+b=0$$

share a common root, find all possible values of

$$\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}.$$

5. Let $f(x)$ be a quadratic polynomial with real coefficients.

   (a) Find a pair of quadratic polynomials $g(x)$ and $h(x)$ with real coefficients such that $f(x)f(x+1) = g(h(x))$.

   (b) Determine if there are infinitely many pairs $g(x)$ and $h(x)$ of quadratic polynomials with real coefficients such that $f(x)f(x+1) = g(h(x))$.

1.15 Basic properties of polynomials

1. [Lagrange's Interpolation Formula] There is a unique second degree polynomial $p(x)$ passing through points $(1,5),(3,8),(6,-7)$. Explain why

$$p(x) = \frac{5(x-3)(x-6)}{(1-3)(1-6)} + \frac{8(x-1)(x-6)}{(3-1)(3-6)} - \frac{7(x-1)(x-3)}{(6-1)(6-3)}.$$

2. (Continuation) In general, let $x_0, x_1, \ldots, x_n$ be distinct real numbers, and let $y_0, y_1, \ldots, y_n$ be arbitrary real numbers. Then there exists a unique polynomial $P(x)$ of degree at most n such that $P(x_i) = y_i$, $i = 0, 1, \ldots, n$. Show that this polynomial is

$$P(x) = \sum_{i=0}^{n} y_i \frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}.$$

3. Let $P(x)$ be a polynomial with leading coefficient 1 and integer coefficients. If u and v are positive integers, where v is not a perfect square, and $u+\sqrt{v}$ is a root of $P(x)$, show that $u-\sqrt{v}$ is also a root of $P(x)$.

4. Let $f(x) = x^4 - 49x^2 - 14x - 1$ and let $g(x) = ax + b$. Find positive integers a and b for which $f(g(x))$ is divisible by $x^2 + 9x + 19$.

5. The polynomial P is a quadratic with integer coefficients. For every positive integer n, the integers $P(n)$ and $P(P(n))$ are relatively prime to n. If $P(3) = 89$, determine with justification the value of $P(10)$?

1.18 Introduction to functional properties (part 3)

1. Given that for any real number x

$$f\left(x - \frac{1}{x}\right) = x^2 + \frac{1}{x^2} + 1$$

find $f(x+1)$ and $f(1)$. What if $f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2} + 1$?

2. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $3f(x) - 4f\left(\frac{1}{x}\right) = x^2$.

3. The graph of the function $f : \mathbb{R} \to \mathbb{R}$ has vertical lines $x = a$ and $x = b$ as lines of reflective symmetry. What other lines of reflective symmetry must have? Determine if f is periodic.

4. Write an identity that says that the graph $y = f(x)$ has a

   (a) period of 12.
   (b) reflective symmetry in the line $x = 12$.
   (c) half-turn symmetry at $(12, 0)$.
   (d) half-turn symmetry at $(1, 2)$.

   Provide an example of a function for each of the above properties. You can't use constant and linear functions in your examples.

5. A certain function $f : [1, \infty) \to \mathbb{R}$ has the properties that $f(3x) = 3f(x)$ and that $f(x) = 1 - |x - 2|$ for $1 \le x \le 3$. Find the smallest x for which $f(x) = f(2013)$.