## 1.16  Starry, Starry Night, E16

1. [IDEA MATH materials] Let $S$ be the set of positive integers $n$ for which the sum of the digits of $n^2$ and the sum of the digits of $(n+1)^2$ are the same.

   (a) Find all integers in $S$ that are less than 50.

   (b) Prove that there are infinitely many integers in $S$.

   <div align="right">Proposed by Ivan Borsenco and Zuming Feng</div>

2. [IMO 2019/5] The Bank of Bath issues coins with an $H$ on one side and a $T$ on the other. Harry has $n$ of these coins arranged in a line from left to right. He repeatedly performs the following operation: if there are exactly $k > 0$ coins showing $H$, then he turns over the $k$th coin from the left; otherwise, all coins show $T$ and he stops. For example, if $n = 3$ the process starting with the configuration $THT$ would be $THT \to HHT \to HTT \to TTT$, which stops after three operations.

   (a) Show that, for each initial configuration, Harry stops after a finite number of operations.

   (b) For each initial configuration $C$, let $L(C)$ be the number of operations before Harry stops. For example, $L(THT) = 3$ and $L(TTT) = 0$. Determine the average value of $L(C)$ over all $2^n$ possible initial configurations $C$.

   <div align="right">Proposed by David Joseph Altizio</div>

3. [ELMO 2013/2] Let $a, b, c$ be positive reals satisfying $a + b + c = \sqrt[7]{a} + \sqrt[7]{b} + \sqrt[7]{c}$. Prove that $a^a b^b c^c \geq 1$.

   <div align="right">Proposed by Evan Chen</div>

4. [USATST 2019/5 simplified version] Tasty is given a circular necklace with $s$ sapphire beads and $t$ turquoise beads such that no two sapphire beads are adjacent to each other. Every minute she removes three consecutive beads that are turquoise, sapphire, turquoise in that order from the necklace. Let $m$ be the maximum number of times Tasty can remove beads from this necklace. Find all possible values of $m$ in terms of $s$ and $t$.

   <div align="right">Proposed by Yannick Yao</div>