1.7 Season 1 Episode 7, 11/1/2015

1. Mark is asked to solve the following problem.

   > Brooks and Avery are running laps around the outdoor track, in the same direction. Brooks completes a lap every 78 seconds while Avery needs 91 seconds for every tour of the track. Brooks (the faster runner) has just passed Avery. How much time will it take for Brooks to overtake Avery again?

   Mark believes that the correct answer is the least common multiple of 78 and 91, which is 546. Note that 546 second is the correct answer to the problem.

   (a) Can you modify the numbers 78 and 91 to show that Mark's reasoning is not correct?

   (b) What is the flaw in Mark's thinking process?

   (c) Can you modify the given question slightly so that Mark's answer is the correct solution (both numerically and logically) to the new question.

2. Consider a (not necessarily regular) convex heptagon $\mathcal{P} = A_1A_2A_3A_4A_5A_6A_7$. There are two natural ways to construct a 7-point star from these seven vertices of the heptagon. One of them is $\mathcal{P}_1 = A_1A_3A_5A_7A_2A_4A_6$, and the other is $\mathcal{P}_2 = A_1A_4A_7A_3A_6A_2A_5$. Determine with justification the sum of vertex angles of the each of $\mathcal{P}_1$ and $\mathcal{P}_2$.

3. A math contest has 100 participants, and consists in finding the answers to three questions. At the end of the contest, one notices that there are a total of exactly 200 correct answers. Prove that one can find 34 participants who have correct answers to the same two questions (and, possibly, to the third question).

4. [IMO 2015, by Merlijn Staps from Netherlands] We say that a finite set $\mathcal{S}$ of points in the plane is *balanced* if, for any two distinct points A and B in $\mathcal{S}$, there is a point C in $\mathcal{S}$ such that $AC = BC$. We say that $\mathcal{S}$ is *center-free* if for any distinct points A, B, and C in $\mathcal{S}$, there is no point P in $\mathcal{S}$ such that $PA = PB = PC$. Find infinitely many positive integers $n \geq 3$ such that there exist a balanced, center-free set consisting of n points.

5. [From Mathematical Puzzles] After the revolution, each of the 66 citizens of a certain country, including the king, has a salary of one dollar. The king can no longer vote, but he does retain the power to suggest changes – namely, redistribution of salaries. Each person's salary must be a whole number of dollars, and the salaries must sum to 66 dollars. Each suggestion is voted on, and carried if there are more votes for than against. Each voter can be counted on to vote *yes* if his salary is to be increased, *no* if decreased, and otherwise not to bother voting.

   The king is both selfish and clever. What is the maximum salary he can obtain for himself, and how long does it take him to get it?