Divisors

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Factoring and fundamental theorem of arithmetic

3.1.1. Suppose that m and n are positive integers such that $75m = n^3$. What is the minimum possible value of $m + n$?

3.1.2. Let a, b, c, d, and e be distinct integers such that

$$(6 - a)(6 - b)(6 - c)(6 - d)(6 - e) = 45.$$

What is $a + b + c + d + e$?

3.1.3. Let p, q and r be distinct prime numbers, where 1 is not considered a prime. What is the smallest positive perfect cube having $n = pq^2r^4$ as a divisor?

3.1.4. Let x and y be positive integers such that $7x^5 = 11y^{13}$. The minimum possible value of x can be written in the form $a^c b^d$, where a, b, c, d are positive integers. Compute $a + b + c + d$.

3.1.5. A positive integer is written on each face of a cube. Each vertex is then assigned the product of the numbers written on the three faces intersecting the vertex. The sum of the numbers assigned to all the vertices is equal to 1001. Find the sum of the numbers written on the faces of the cube.

3.1.6. Find the largest integer n such that 2^n divides $3^{1024} - 1$. (For a prime p we say that p^k *fully divides* n and write $p^k \| n$ if k is the greatest positive integer such that $p^k | n$.)

3.1.7. Let $N = 69^5 + 5 \cdot 69^4 + 10 \cdot 69^3 + 10 \cdot 69^2 + 5 \cdot 69 + 1$. How many positive integers are factors of N?

3.1.8. If

$$\frac{m}{n} = \frac{1}{0!10!} + \frac{1}{1!9!} + \frac{1}{2!8!} + \frac{1}{3!7!} + \frac{1}{4!6!} + \frac{1}{5!5!}$$

where m and n are relatively prime positive integers. Compute the sum of the prime divisors of m plus the sum of the prime divisors of n.

3.1.9. Given that

$$\frac{1}{2!17!} + \frac{1}{3!16!} + \frac{1}{4!15!} + \frac{1}{5!14!} + \frac{1}{6!13!} + \frac{1}{7!12!} + \frac{1}{8!11!} + \frac{1}{9!10!} = \frac{N}{1!18!}.$$

find the greatest integer that is less than $N/100$.

3.1.10. The number 27000001 has exactly four prime factors. Find their sum.

3.1.11. Prove that for any integer n greater than 1, the number $n^5 + n^4 + 1$ is composite.

3.1.12. Four positive integers a, b, c, d satisfy

$$ab + a + b = 524,$$
$$bc + c + b = 146,$$
$$cd + c + d = 104.$$

Find all possible values of $a - d$.

3.1.13. Call a number *prime looking* if it is composite but not divisible by 2, 3, or 5. The three smallest prime-looking numbers are 49, 77, and 91. There are 168 prime numbers less than 1000. How many prime-looking numbers are there less than 1000?

3.1.14. Find the largest divisor of 1001001001 that does not exceed 10000.

The number of divisors

3.1.15. Compute the probability that a randomly chosen positive divisor of 10^{99} is an integer multiple of 10^{88}.

3.1.16. One of the legs of a right triangle is $2^{34} \cdot 3^{21}$ units long. Both the other leg and the hypotenuse are of integer lengths. How many such triangles are there? Determine the number incongruent right triangles with one its leg equal to $2^{34}3^{21}$ and the lengths of the other two sides are also integers.

3.1.17. A positive integer n has 60 divisors and $7n$ has 80 divisors. What is the greatest integer k such that 7^k divides n.

3.1.18. Let n be the smallest positive integer that is a multiple of 75 and has exactly 75 positive integral divisors, including 1 and itself. Find $\frac{n}{75}$.

3.1.19. For how many values of k is 12^{12} the least common multiple of the positive integers 6^6, 8^8, and k?

3.1.20. How many positive integers have exactly three proper divisors, each of which is less than 50? (A *proper divisor* of a positive integer n is a positive integer divisor other than n.)

3.1.21. If n is a positive integer such that $2n$ has 28 positive divisors and $3n$ has 30 positive divisors, then how many positive divisors does $6n$ have?

3.1.22. Find the smallest positive integers that has exactly 30 positive integer divisors.

3.1.23. Which 3-digit number has the greatest number of different factors?

3.1.24. Determine the number of ordered pairs of positive integers (a, b) such that the least common multiple of a and b is $2^3 5^7 11^{13}$.

3.1.25. Let $n = 2^{31}3^{19}$. How many positive integer divisors of n^2 are less than n but do not divide n?

3.1.26. Find all primes p such that the number $p^2 + 11$ has exactly six different divisors (including 1 and the number itself).

The sum and product of divisors

3.1.27. Find the sum of even positive divisors of 100000.

3.1.28. Determine the product of distinct positive integer divisors of 420^4.

3.1.29. Given a positive integer n, let $p(n)$ be the product of the nonzero digits of n. (If n has only one digit, then $p(n)$ is equal to that digit.) Let

$$S = p(1) + p(2) + \cdots + p(999).$$

What is the largest prime factor of S?

3.1.30. Compute the sum of all numbers of the form a/b, where a and b are relatively prime positive divisors of 27000.

3.1.31. For how many three-element sets of positive integers $\{a, b, c\}$ is it true that $a \times b \times c = 2310$.

3.1.32. How many positive integer divisors of 2004^{2004} are divisible by exactly 2004 positive integers?

3.1.33. Let S be the sum of all numbers of the form a/b, where a and b are relatively prime positive divisors of 1000. What is the greatest integer that does not exceed $S/10$?

3.1.34. For each even positive integer x, let $g(x)$ denote the greatest power of 2 that divides x. For example, $g(20) = 4$ and $g(16) = 16$. For each positive integer n, let

$$S_n = \sum_{k=1}^{2^{n-1}} g(2k).$$

Find the greatest integer n less than 1000 such that S_n is a perfect square.

3.1.35. For positive integer n, let $\tau(n)$ denote the number of distinct positive integer divisors of n, including 1 and n. For example, $\tau(1) = 1$ and $\tau(3) = 2$. Define $S(n)$ by

$$S(n) = \tau(1) + \tau(2) + \cdots + \tau(n).$$

Let a denote the number of positive integers $n \leq 2005$ with $S(n)$ odd, and let b denote the number of positive integers $n \leq 2005$ with $S(n)$ even. Find $|a - b|$.