Number Sense

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1. Letters A, B, C, and D represent four different digits selected from $0, 1, 2, \ldots, 9$. If $(A+B)/(C+D)$ is an integer that is as large as possible, what is the value of $A+B$?

2. Consider the sequence $1, -2, 3, -4, 5, -6, \ldots$, whose n^{th} term is $(-1)^{n+1} \cdot n$. What is the average of the first 200 terms of the sequence?

3. Find all real numbers x such that $|x-2| + |x-3| = 1$.

4. Find the number of positive integers k for which the equation $kx - 12 = 3k$ has an integer solution for x.

5. How many positive integers less than 50 have an odd number of positive divisors?

6. Four girls – Mary, Alina, Tina, and Hanna – sang songs in a concert as trios, with one girl sitting out each time. Hanna sang 7 songs, which was more than any other girls, and Mary sang 4 songs, which is fewer than any other girl. How many songs did these trios sing?

7. Which of the numbers 25, 33, 52, 66, and 154 is the average of the other four numbers?

8. In the sixth, seventh, eighth, and ninth basketball games of the season, a player scored 23, 14, 11, and 20 points, respectively. Her points-per-game average was higher after nine games than it was after the first five games. If her average after ten games was greater than 18, what is the least number of points she could have scored in the tenth game?

9. There exist unique positive integers x and y that satisfy the equation $x^2 + 84x + 2008 = y^2$. Find $x + y$.

10. How many pair of positive integers (a, b) with $a + b \le 100$ satisfy the equation
$$\frac{a + b^{-1}}{a^{-1} + b} = 13?$$

11. The 2-digit integers from 19 to 92 are written consecutively to form the large integer
$$N = 19202122\ldots909192.$$
If 3^k is the highest power of 3 that is a factor of N, find k.

12. The increasing sequence of positive integers $a_1, a_2, a_3, \ldots$ has the property that $a_{n+2} = a_n + a_{n+1}$ for all $n \ge 1$. If $a_7 = 120$, find a_8.

13. Let $N = 100^2 + 99^2 - 98^2 - 97^2 + 96^2 + \cdots + 4^2 + 3^2 - 2^2 - 1^2$, where the additions and subtractions alternate in pairs. Find the remainder when N is divided by 1000.

14. What is the size of the largest subset, S, of $\{1, 2, 3, \ldots, 50\}$ such that no pair of distinct elements of S has a sum divisible by 7?

15. A subset of the integers $1, 2, \ldots, 100$ has the property that none of its members is 3 times another. What is the largest number of members such a subset can have?

16. For how many integers N between 1 and 1999 is the improper fraction $\frac{N^2+7}{N+4}$ not in lowest term?

17. Three cards, each with a positive integer written on it, are lying face-down on a table. Casey, Stacy, and Tracy are told that

    (a) the numbers are all different,

    (b) they sum to 13, and

    (c) they are in increasing order, left to right.

    First, Casey looks at the number on the leftmost card and says, "I dont have enough information to determine the other two numbers." Then Tracy looks at the number on the rightmost card and says, "I dont have enough information to determine the other two numbers." Finally, Stacy looks at the number on the middle card and says, "I dont have enough information to determine the other two numbers." Assume that each person knows that the other two reason perfectly and hears their comments. What number is on the middle card?

18. Let $1, 4, \ldots$ and $9, 16, \ldots$ be two arithmetic progressions. The set S is the union of the first 2004 terms of each sequence. How many distinct numbers are in S?

19. A block of cheese in the shape of a rectangular solid measures 10 cm by 13 cm by 14 cm. Ten slices are cut from the cheese. Each slice has a width of 1 cm and is cut parallel to one face of the cheese. The individual slices are not necessarily parallel to each other. What is the maximum possible volume in cubic cm of the remaining block of cheese after ten slice have been cut off?

20. First a is chosen at random from the set $\{1, 2, 3, \ldots, 99, 100\}$, and then b is chosen randomly form the same set. What is the probability that the integer $3^a + 7^b$ has units digit 8?

21. Let x and y be positive integers such that $7x^5 = 11y^{13}$. The minimum possible value of x can be written in the form $a^c b^d$, where a, b, c, d are positive integers. Compute $a + b + c + d$.

22. Which one of the following integers: (A) 1,627,384,950; (B) 2,345,678,910; (C) 3,579,111,300; (D) 4,692,581,470; (E) 5,815,937,260 can be expressed as the sum of 100 consecutive positive integers?

23. A triangular array of numbers has a first row consisting of the odd integers $1, 3, 5, \ldots, 99$ in increasing order. Each row below the first has one fewer entry than the row above it, and the bottom row has a single entry. Each entry in any row after the top row equals the sum of the two entries diagonally above it in the row immediately above it. How many entries in the array are multiples of 67?

$$\begin{array}{ccccccccccc} 1 & & 3 & & 5 & & \cdots & & 97 & & 99 \\ & 4 & & 8 & & 12 & & \cdots & & 196 & \\ & & & & & \vdots & & & & & \end{array}$$

24. The sequence $\{a_n\}$ is defined by

$$a_0 = 1,\ a_1 = 1,\ \text{and } a_n = a_{n-1} + \frac{a_{n-1}^2}{a_{n-2}} \text{ for } n \geq 2.$$

The sequence $\{b_n\}$ is defined by

$$b_0 = 1,\ b_1 = 3,\ \text{and } b_n = b_{n-1} + \frac{b_{n-1}^2}{b_{n-2}} \text{ for } n \geq 2.$$

Find $\frac{b_{32}}{a_{32}}$.

25. Let $S = \{1, 2, 3, \dots, 24, 25\}$. Compute the number of elements in the largest subset of S such that no two elements in the subset differ by the square of an integer.

26. Ten people form a circle. Each picks a number and tells it to the two neighbors adjacent to him in the circle. Then each person computes and announces the average of the numbers of his two neighbors. In clockwise order, the averages announced by each person are $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$. What is the number picked by the person who announced the average 6?

27. There exist r unique nonnegative integers $n_1 > n_2 > \cdots > n_r$, and r unique integers a_k $(1 \leq k \leq r)$ with each a_k either 1 or -1 such that

$$a_1 \cdot 3^{n_1} + a_2 \cdot 3^{n_2} + \cdots + a_r \cdot 3^{n_r} = 2008.$$

Find $n_1 + n_2 + \cdots + n_r$.

28. A team wins 3 games, then loses 1, then wins 3 and loses 2, then wins 3 and loses 3, and so on, each time winning 3 games before losing one more than before. If N is the number of games played, find the least value of N such that the percentage of wins is below 25%.

29. Four positive integers a, b, c, and d satisfy

$$\begin{aligned} ab + a + b &= 524, \\ bc + b + c &= 146, \\ cd + c + d &= 104. \end{aligned}$$

Find all the possible values of $a - d$.

30. Ashley, Betty, Carlos, Dick, and Elgin went shopping. Each had a whole number of dollars to spend, and together they had \$56. The absolute difference between the amounts Ashley and Betty had to spend was \$19. The absolute difference between the amounts Betty and Carlos had was \$7, between Carlos and Dick was \$5, between Dick and Elgin was \$4, and between Elgin and Ashley was \$11. How much did Elgin have?

31. Let $x_1, x_2, \dots, x_n$ be a sequence of integers such that

    (i) $-1 \leq x_i \leq 2$, for $i = 1, 2, 3, \dots, n$;

    (ii) $x_1 + x_2 + \cdots + x_n = 19$; and

    (iii) $x_1^2 + x_2^2 + \cdots + x_n^2 = 99$.

    Let m and M be the minimal and maximal possible values of $x_1^3 + x_2^3 + \cdots + x_n^3$, respectively. Compute $M + m$.

32. Several positive integers are given, not necessarily all different. Their sum is 2003. Suppose that n_1 of the given numbers are equal to one, n_2 of them are equal to two, and so on, n_{2003} of them are equal to 2003. Find the largest possible value of $n_2 + 2n_3 + 3n_4 + \cdots + 2002n_{2003}$.

33. In the following 3×3 array of positive integers, the products of the entries of each row, column, and diagonal are the same. What is the sum of all the possible values of g?

$$\begin{matrix} 50 & b & c \\ d & e & f \\ g & h & 2 \end{matrix}$$

34. In a classroom, 34 students are seated in five rows of seven chairs. The place at the center of the room is unoccupied. A teacher decides to reassign the seats such that *each* student will occupy a chair adjacent to his/her present one (that is, move one desk forward, back, left or right). In how many ways can this reassignment be made?

35. Call a positive real number *special* if it has a decimal representation that consists entirely of digits 0 and 7. For example, $\frac{700}{99} = 7.\overline{07}$ and 77.007 are special numbers. What is the smallest n such that 1 can be written as a sum of n special numbers?