Ptolemy's Theorem

Oleg Golberg

1 Introduction

Theorem 1 (Ptolemy's Theorem)**.** If $ABCD$ is a cyclic quadrilateral, then $AB \cdot CD + AD \cdot BC = AC \cdot BD$.

Theorem 2 (Ptolemy's Inequality)**.** If A, B, C, D are four points in the plane, then $AB \cdot CD + AD \cdot BC \geq AC \cdot BD$. Equality is achieved iff $ABCD$ is a (possibly degenerate) cyclic quadrilateral.

Problem 1. Let $ABCD$ be a cyclic quadrilateral. Construct point E on diagonal BD such that $\angle AED = \angle ABC$. Find two pairs of similar triangles on the diagram and deduce Ptolemy's Theorem.

Problem 2. Let a, b, c, d be complex numbers. Prove that

$$(a-b)(c-d) + (d-a)(c-b) = (a-c)(b-d).$$

Deduce Ptolemy's Inequality.

Problem 3. Let A, B, C, D be four points in the plane. Let $B^\star, C^\star, D^\star$ be the images of B, C, D respectively under the inversion with center A and radius 1. Find the distances between $B^\star, C^\star, D^\star$ in terms of the distances between A, B, C, D. Deduce Ptolemy's Inequality.

2 Problems

Problem 4. Let $\alpha = \frac{\pi}{7}$. Prove that

$$\frac{1}{\sin \alpha} = \frac{1}{\sin 2\alpha} + \frac{1}{\sin 3\alpha}.$$

Problem 5. Let $ABCD$ be a cyclic quadrilateral. Prove that

$$\frac{AC}{BD} = \frac{AB \cdot AD + CB \cdot CD}{BA \cdot BC + DA \cdot DC}.$$

Problem 6. Let d_a, d_b, d_c be the distances from the circumcenter of an acute triangle to its sides and let R and r be its circumradius and inradius respectively. Prove that $d_a + d_b + d_c = R + r$.

Problem 7. The bisector of angle A of triangle ABC meets its circumcircle at D. Prove that $AB + AC \leq 2AD$.

Problem 8. Point P is chosen on the arc CD of the circumcircle of a square $ABCD$. Prove that $PA + PC = \sqrt{2}PB$.

Problem 9. Let $ABCD$ be a parallelogram. A circle passes through A and meets segments AB, AC, AD at P, Q, R respectively. Prove that $AP \cdot AB + AR \cdot AD = AQ \cdot AC$.

Problem 10. (a) Point D is chosen on the arc BC of the circumcircle of an equilateral triangle ABC. Prove that $DA = DB + DC$.

(b) Point A is chosen on the arc A_1A_{2n+1} of the circumcircle of a regular $(2n+1)$-gon $A_1A_2\cdots A_{2n+1}$. Prove that

$$AA_1 + AA_3 + \cdots + AA_{2n+1} = AA_2 + AA_4 + \cdots + AA_{2n}.$$

Problem 11. Circles of radii x and y are tangent externally to a circle of radius R. The distance between the points of tangency is a. Find the length of a common external tangent to the circles.

Theorem 3 (Casey's Theorem)**.** Circles $\alpha, \beta, \gamma, \delta$ are externally tangent to a fifth circle at A, B, C, D respectively, and $ABCD$ is a convex quadrilateral. Let $t_{\alpha\beta}$ be the length of a common external tangent to α and β. Define $t_{\beta\gamma}$ etc. similarly. Then

$$t_{\alpha\beta}t_{\gamma\delta} + t_{\beta\gamma}t_{\delta\alpha} = t_{\alpha\gamma}t_{\beta\delta}.$$

Homework Problem

Problem 1. (a) Let ω_1, ω_2 be disjoint circles and let A be a point outside both of them. Let t_1, t_2 be the lengths of tangents from A to ω_1, ω_2 respectively and let t be the length of a common external tangent to ω_1, ω_2. Let $\omega_1^\star, \omega_2^\star$ be the images of ω_1, ω_2 respectively under the inversion with center A and radius 1. Find the length of a common external tangent to ω_1, ω_2 in terms of t, t_1, t_2.

(b) Devise a proof of Casey's theorem that mimics the "inversion proof" of Ptolemy's theorem.