2.1. Explain the existence of the circumcenter \( O \), the incenter \( I \), and the excenters \( I_A, I_B, \) and \( I_C \) of triangle \( ABC \).

2.2. Explain the existence of the orthocenter \( H \) of triangle \( ABC \).

2.3. Let \( AB \) be a segment. Points \( X \) and \( Y \) do not lie on line \( AB \). Point \( Z \) lies on line \( AB \). Then \( X, Y, Z \) are collinear (that is, they lie on a line) if and only if \( \frac{AXY}{BXY} = \frac{AZ}{BZ} \).

2.4. Explain the existence of the centroid \( G \) of triangle \( ABC \) by establishing the fact that triangles \( ABG, BCG, \) and \( CAG \) have the same area.

2.5. The three medians cut the triangle into 6 smaller triangles with equal area. The centroid of the triangle lies 2/3 along way (from the vertex to the opposite midpoint) on each median.

2.6. [Euler] Prove that the orthocenter, circumcenter, and centroid of a triangle lie on a line. This line is called the Euler line of the triangle.

2.7. Let \( ABC \) be a triangle with circumcircle \( \omega \). Let \( O, G, H, I, I_A \) denote its circumcenter, centroid, orthocenter, incenter, excenter opposite \( A \), respectively. Points \( M \) and \( H_A \) lie on \( \hat{BC} \) (not including \( A \)) such that \( \hat{BM} = \hat{MC} \) and \( AH_A \perp BC \). Let \( A_1 \) be the midpoint of side \( BC \). The following are true.

(a) points \( O, A_1, M \) are collinear;

(b) \( H \) and \( H_A \) are symmetric across the line \( BC \);

(c) \( G \) lies on segment \( OH \) with \( OG \) with \( 2OG = GH \), and \( G \) is the intersection of segments \( AA_1 \) and \( OH \);

(d) points \( A, I, M, I_A \) are collinear;

(e) points \( B, C, I, I_A \) lie on a circle centered at \( M \).
2.8. Triangle $ABC$ is inscribed in circle $\omega$. Let $A_1$ be the midpoint of arc $BC$ (not containing $A$). Define points $B_1$ and $C_1$ analogously. Show that the incenter of triangle $ABC$ is the orthocenter of triangle $A_1B_1C_1$.

2.9. The incircles of triangle $ABC$ is tangent to sides $BC, CA, AB$ at $D, E, F$, respectively. Let $I_A, I_B, I_C$ be the incenters of triangles $AEF, BDF, CDE$, respectively. Prove that lines $I_AD, I_BE, I_CF$ are concurrent.

2.10. Let $ABC$ be a triangle with excenters $I_A, I_B,$ and $I_C$.

(a) Prove that the incenter of triangle $ABC$ is the orthocenter of triangle $I_AI_BI_C$.

(b) Prove that triangle $I_AI_BI_C$ is acute.

(c) Prove that there is a point $O$ such that $I_AO \perp BC, I_BO \perp CA, I_CO \perp AB$.

2.11. Let $ABC$ be an acute-angled scalene triangle, and let $H, I$, and $O$ be its orthocenter, incenter, and circumcenter, respectively. Circle $\omega$ passes through points $H, I,$ and $O$. Prove that if one of the vertices of triangle $ABC$ lies on circle $\omega$, then there is one more vertex lies on $\omega$.

2.12. In triangle $ABC$, $\angle BAC = 120^\circ$. The angles bisectors of angles $A, B,$ and $C$ meet the opposite sides at $D, E,$ and $F$, respectively. Compute $\angle EDF$.

2.13. In triangle $ABC$, $AB = 14$, $BC = 16$, and $CA = 26$. Let $M$ be the midpoint of side $BC$, and let $D$ be a point on segment $BC$ such that $AD$ bisects $\angle BAC$. Compute $PM$, where $P$ is the foot of perpendicular from $B$ to line $AD$.

2.14. Given a circle $\omega$ and two fixed points $A$ and $B$ on the circle. Assume that there is a point $C$ on $\omega$ such that $AC + BC = 2AB$.

(a) Show that the line passing through the incenter and the centroid of the triangle is parallel to one the side of the triangle.

(b) How to construct point $C$ with a compass and a straightedge.