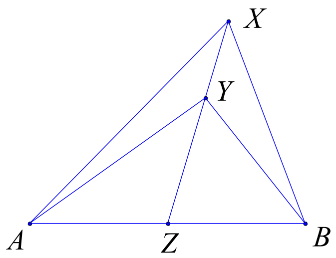
Centers of triangles, a beginner's tour

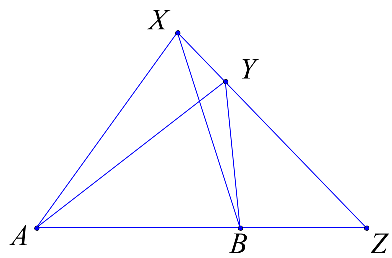
September 2007

2.1. Explain the existence of the circumcenter O, the incenter I, and the excenters I_A, I_B, and I_C of triangle ABC.

2.2. Explain the existence of the orthocenter H of triangle ABC.

2.3. Let AB be a segment. Points X and Y do not lie on line AB. Point Z lies on line AB. Then X, Y, Z are **collinear** (that is, they lie on a line) if and only if $[AXY]/[BXY] = AZ/BZ$.





2.4. Explain the existence of the centroid G of triangle ABC by establishing the fact that triangles ABG, BCG, and CAG have the same area.

2.5. The three medians cut the triangle into 6 smaller triangles with equal area. The centroid of the triangle lies 2/3 along way (from the vertex to the opposite midpoint) on each median.

2.6. [Euler] Prove that the orthocenter, circumcenter, and centroid of a triangle lie on a line. This line is called the **Euler line** of the triangle.

2.7. Let ABC be a triangle with circumcircle ω. Let O, G, H, I, I_A denote its circumcenter, centroid, orthocenter, incenter, excenter opposite A, respectively. Points M and H_A lie on $\widehat{BC}$ (not including A) such that $\widehat{BM} = \widehat{MC}$ and $AH_A \perp BC$. Let A_1 be the midpoint of side BC. The following are true.

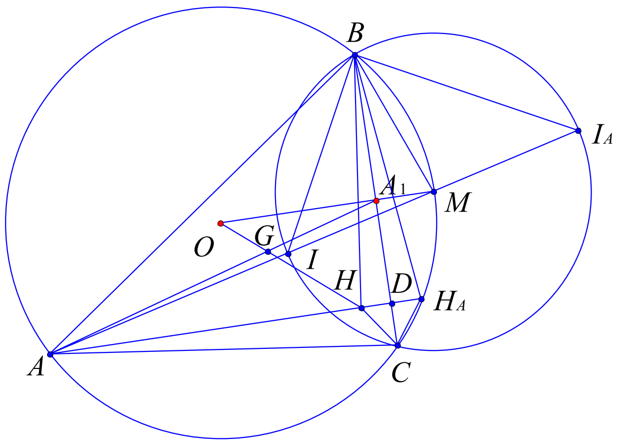
(a) points O, A_1, M are collinear;

(b) H and H_A are symmetric across the line BC;

(c) G lies on segment OH with OG with $2OG = GH$, and G is the intersection of segments AA_1 and OH;

(d) points A, I, M, I_A are collinear;

(e) points B, C, I, I_A lie on a circle centered at M.



2.8. Triangle ABC is inscribed in circle ω. Let A_1 be the midpoint of arc $\widehat{BC}$ (not containing A). Define points B_1 and C_1 analogously. Show that the incenter of triangle ABC is the orthocenter of triangle $A_1B_1C_1$.

2.9. The incircles of triangle ABC is tangent to sides BC, CA, AB at D, E, F, respectively. Let I_A, I_B, I_C be the incenters of triangles AEF, BDF, CDE, respectively. Prove that lines I_AD, I_BE, I_CF are concurrent.

2.10. Let ABC be a triangle with excenters I_A, I_B, and I_C.

(a) Prove that the incenter of triangle ABC is the orthocenter of triangle $I_AI_BI_C$.

(b) Prove that triangle $I_AI_BI_C$ is acute.

(c) Prove that there is a point O such that $I_AO \perp BC$, $I_BO \perp CA$, $I_CO \perp AB$.

2.11. Let ABC be an acute-angled scalene triangle, and let H, I, and O be its orthocenter, incenter, and circumcenter, respectively. Circle ω passes through points H, I, and O. Prove that if one of the vertices of triangle ABC lies on circle ω, then there is one more vertex lies on ω.

2.12. In triangle ABC, $\angle BAC = 120°$. The angles bisectors of angles A, B, and C meet the opposite sides at D, E, and F, respectively. Compute $\angle EDF$.

2.13. In triangle ABC, $AB = 14$, $BC = 16$, and $CA = 26$. Let M be the midpoint of side BC, and let D be a point on segment BC such that AD bisects $\angle BAC$. Compute PM, where P is the foot of perpendicular from B to line AD.

2.14. Given a circle ω and two fixed points A and B on the circle. Assume that there is a point C on ω such that $AC + BC = 2AB$.

(a) Show that the line passing through the incenter and the centroid of the triangle is parallel to one the side of the triangle.

(b) How to construct point C with a compass and a straightedge.