Special angles

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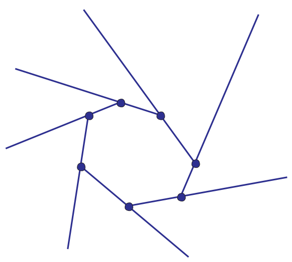
Sentry theorem

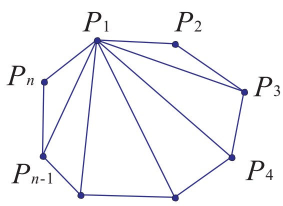
Example 1.1. Alex walks along the boundary of a n-sided plot of land, writing down the number of degrees turned at each corner. What is the sum of these n numbers for $n = 3, 4, 5, \dots$?

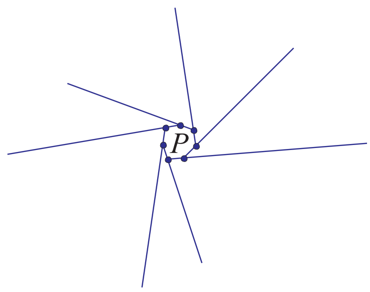
[Sentry theorem] The sum of the exterior angles (one per vertex) of any polygon is 360 degrees. The sum of interior angles of a n-sided convex polygon is $(n-2) \cdot 180°$.

Solution: Since Alex made a $360°$ rotation in direction, the sum is $360°$, independent to n. This explains why the sum of the exterior angles (one per vertex) of any polygon is 360 degrees.

An n-sided convex polygon $P_1P_2 \dots P_n$ can be dissected into $n-2$ triangles with diagonals P_1P_3, P_1P_4, $\dots$, P_1P_{n-1}. Hence the sum of the interior angles is $180° \cdot (n-2)$. Because the sum of exterior and interior angles at each vertex is equal to $180°$, we can conclude again that sum of the exterior angles is equal to $180° \cdot n - 180° \cdot (n-2) = 360°$. ■



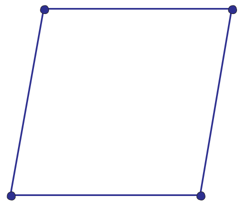


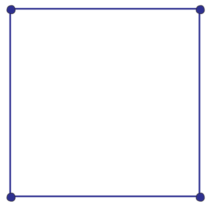
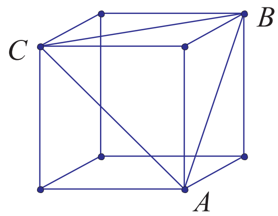


Example 1.2. The sides of a polygon are cyclically extended to form *rays*, creating one exterior angle at each vertex. Viewed from a great distance, what theorem does this figure illustrate?

Solution: Since rays extends forever, rays remain as rays when viewed from a great distance. On the other hand, polygon becomes a point when viewed from a great distance. Thus, when viewed from a great distance, our diagram becomes lines drawing from a common point, implying that the exteriors angles add up to $360°$. This confirms the sentry theorem. ■

A polygon is *equilateral* if its sides have the same length. A polygon is *equiangular* if its interior angles are the same size. Clearly, a polygon is equiangular if its exterior angles are the same size. For a triangle, equilateral is equivalent to equiangular. For polygons with more than 3 sides, these two concepts are not equivalent anymore. For example, rhombus is equilateral but not necessarily equiangular. On the other hand, a rectangle is equiangular but not necessarily equilateral. A polygon that is both equilateral and equiangular is called *regular*. Square is the regular quadrilateral.



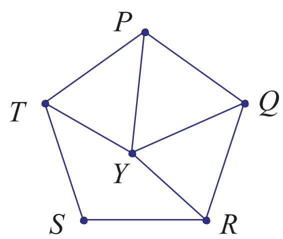



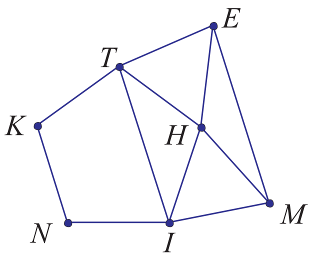
Example 1.3. Find the angle formed by two face diagonals that intersect at a vertex of the cube.

Solution: The answer is 60°. Let AB and AC denote the two face diagonals. Then BC is also a face diagonal. Thus, ABC form an equilateral triangle, and $\angle ABC = \angle BCA = \angle CAB = 60^\circ$. ■

Example 1.4. Mark Y inside regular pentagon $PQRST$, so that PQY is equilateral. Is RYT straight? Explain.

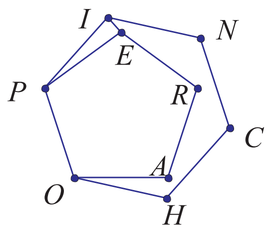
Solution: By the sentry theorem, the exterior angles of a regular pentagon is 72°. Thus, the interior angles of a regular pentagon is 108°. We have $\angle TPY = 108^\circ - 60^\circ = 48^\circ$. Triangle TPY is isosceles with $PT = PQ = PY$, implying that $\angle PYT = (180^\circ - 48^\circ)/2 = 66^\circ$. By symmetry, $\angle QYR = 66^\circ$. It follows that $\angle TPY + \angle PYQ + \angle QYR = 192^\circ \neq 180^\circ$, and so R, Y, T do not lie on a line. ■

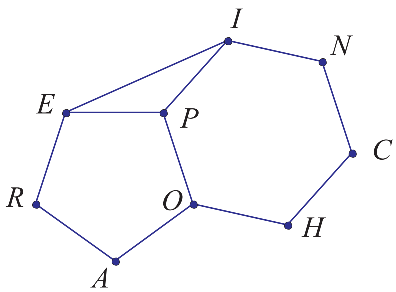




Example 1.5. Equilateral triangles THE and HIM are attached to the outside of regular pentagon $THINK$. Is quadrilateral $TIME$ a parallelogram? Justify your answer.

Solution: As shown in the solution of Example 1.4, we have $\angle THI = 108^\circ$ and $\angle HIT = \angle HTI = 36^\circ$. It follows that $\angle ETI = \angle MIT = 60^\circ + 36^\circ = 96^\circ$. Since $\angle ETI + \angle MIT > 180^\circ$, TE an MI are not parallel to each other, and so $TIME$ is not a parallelogram. ■





Example 1.6. Let $CHOPIN$ be a regular hexagon, and let $OPERA$ be a regular pentagon. Find all possible values of measure of $\angle PIE$.

Solution: The values are $84°$ and $24°$.

By the sentry theorem, the exterior of a regular hexagon is $60°$. implying that the interior angle of a regular hexagon is $120°$.

If E lies inside $CHOPIN$, then $\angle EPI = 120° - 108° = 12°$. Note that triangle PIE isosceles with $IP = EP$. Hence $\angle PIE = (180° - 12°)/2 = 84°$.

If E lies outside $CHOPIN$, then $\angle EPI = 360° - 120° - 108° = 132°$. Note that triangle PIE isosceles with $IP = EP$. Hence $\angle PIE = (180° - 132°)/24°$. ■

Example 1.7. A regular n-sided polygon has interior angles of m degrees each. Express m in terms of n. For how many of these regular examples is m a whole number?

Solution: Each exterior angle of the polygon is $(180 - m)°$. By the sentry theorem, $(180 - m)n = 360$. Hence n is a divisor of $360 = 2^3 \cdot 3^2 \cdot 5$, which has $(3 + 1)(2 + 1)(1 + 1) = 24$ divisors. Since a polygon has at least 3 sides, $n \neq 1$ and $n \neq 2$. Thus, there are $24 - 2 = 22$ such regular polygons.

■

Exercises A

2.1. For what positive integer n is the following statement true: An n-sided polygon is equiangular if and only if it is equilateral.

2.2. Suppose that $ABCD$ is a square, and that CDP is an equilateral triangle, with P *outside* the square. What is the size of angle PAD? What if P is *inside* the square?

2.3. Given regular hexagon $BAGELS$, show that SEA is an equilateral triangle and that segments BA and AE are perpendicular.

2.4. Let $ABCDE$ be a regular pentagon.

(a) Draw diagonal AC. What are the sizes of the angles of triangle ABC? Prove that segments AC and DE are parallel.

(b) Diagonals AC and BD of regular pentagon $ABCDE$ intersect at H. Decide whether or not $AHDE$ is a rhombus, and give your reasons.

(c) Inside the pentagon is marked point P so that triangle ABP is equilateral. Decide whether or not quadrilateral $ABCP$ is a parallelogram, and give your reasons.

2.5. Given square $ABCD$, let P and Q be the points outside the square that make triangles CDP and BCQ equilateral. Prove that triangle APQ is also equilateral. What if $ABCD$ is a rectangle? What if $ABCD$ is a parallelogram?

2.6. Let's work with equilateral triangles.

(a) The sides of an equilateral triangle are 12 cm long. How long is an altitude of this triangle? What are the angles of a right triangle created by drawing an altitude? How does the short side of this right triangle compare with the other two sides?

(b) The altitudes of an equilateral triangle all have length 12 cm. How long are its sides?

(c) If the area of an equilateral triangle is $100\sqrt{3}$ square inches, how long are its sides?

2.7. Find the side of the largest square that can be drawn inside a 12-inch equilateral triangle, one side of the square aligned with one side of the triangle.

2.8. A park is in the shape of a regular hexagon 2km on a side. Starting at a corner, Alice walks along the perimeter of the park for a distance of 5km. How many kilometers is she from her starting point?

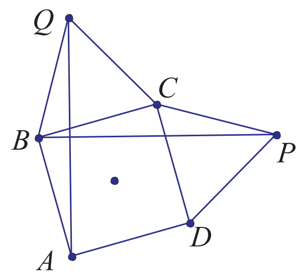
2.9. Three non-overlapping regular plane polygons all have sides of length 1. The polygons meet at a point A in such a way that the sum of the three interior angles at A is 360°. Thus the three polygons form a new polygon $\mathcal{P}$ (not necessarily convex) with A as an interior point. Among the three polygons, one is a square and one is a pentagon. Find the perimeter of $\mathcal{P}$.

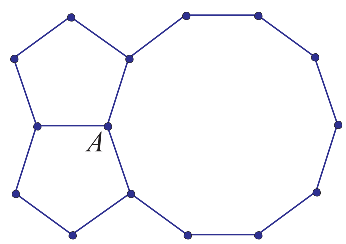
2.10. Find an example of an equilateral hexagon, with its vertices being lattice points, whose sides are all $\sqrt{13}$ units long. Give coordinates for all six points. Compute the area of the hexagon.

Connected regular polygons

Example 3.1. Given square $ABCD$, let P and Q be the points outside the square that make triangles CDP and BCQ equilateral. Segments AQ and BP intersect at T. Find angle ATP.

Solution: The answer is 90°. Note that triangles ABQ and BCP are congruent. Furthermore, we can rotate triangle ABQ to obtain triangle BCP. More precisely, consider the rotation $\mathbf{R}$ of 90° centered at the center of the square $ABCD$ that taking A to B. Then $\mathbf{R}(B) = C$, $\mathbf{R}(C) = D$, and $\mathbf{R}(BCQ) = CDP$. Thus, $\mathbf{R}(ABQ) = BCP$. In particular, $\mathbf{R}(AQ) = BP$, implying that $AQ \perp BP$. ■





Example 3.2. [AHSME 1999] Three non-overlapping regular plane polygons all have sides of length 1. The polygons meet at a point A in such a way that the sum of the three interior angles at A is 360°. Thus the three polygons form a new polygon $\mathcal{P}$ (not necessarily convex) with A as an interior point. Among the three polygons, at least two of them are congruent. Find the possible values of the perimeter of $\mathcal{P}$.

Solution: Suppose that we have two n-sided polygons, with interior angle equal to $\left(180 - \frac{360}{n}\right)^\circ$, and one m-sided polygon, with interior angle equal to $\left(180 - \frac{360}{m}\right)^\circ$. Thus, we have

$$2 \cdot \left(180 - \frac{360}{n}\right)^\circ + \left(180 - \frac{360}{m}\right)^\circ = 360^\circ \quad \text{or} \quad \frac{2}{m} + \frac{4}{n} = 1. \qquad (*)$$

We can finish in two way.

- *First approach* Note that $m \geq 3$. Thus, $\frac{4}{n} = 1 - \frac{2}{m} \geq \frac{1}{3}$, and so $n \leq 12$. Hence $3 \leq n \leq 12$. it is not difficult to find $(m, n) = (3, 12), (4, 8), (6, 6), (10, 5)$.
- *Second approach* Clearing the denominator and regrouping gives $mn - 4m - 2n = 0$ or $(m - 2)(n - 4) = 8$. Thus, the possible choices for $(m - 2, n - 4)$ are $(1, 8), (2, 4), (4, 2), (8, 1)$, and so $(m, n) = (3, 12), (4, 8), (6, 6), (10, 5)$.

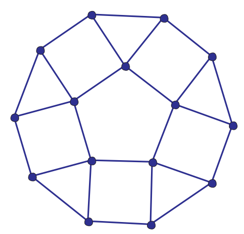
The perimeter of $\mathcal{P}$ is equal to $n + n + m - 6$, and its possible values are $21, 14, 12$. ■

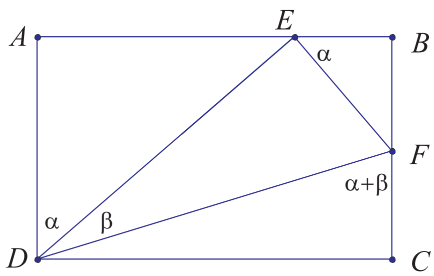
Note: What if the condition "at least two of them are congruent" is removed?

Example 3.3. Suppose that $DRONE$ is a regular pentagon, and that $DRUM$, $ROCK$, $ONLY$, $NEAP$, and $EDIT$ are squares attached to the outside of the pentagon. Is the decagon $ITAPLYCKUM$ equiangular? Is it equilateral?

Solution: The decagon $ITAPLYCKUM$ is equiangular but not equilateral.

Note that $\angle DRO = 108^\circ$ and $\angle UKR = 72^\circ$. In isosceles triangle KRU, $\angle UKR = \angle RUK = 54^\circ$ and $RK = RU \neq UK$. On the other hand, in square $DRUM$, $MU = UR$. It is easy to see that $UK = CY = LP = AT = IM \neq MU = KC = YL = PA = TI$. (It is certainly equiangular.) ■





Example 3.4. For two angles α (Greek "alpha") and β (Greek "beta") with $0^\circ < \alpha, \beta, \alpha + \beta < 90^\circ$, it is not difficult to note that the trigonometric functions do not satisfy the additive distributive law; that is, identities such as $\sin(\alpha+\beta) = \sin\alpha + \sin\beta$ and $\cos(\alpha+\beta) = \cos\alpha + \cos\beta$ are not true. For example, setting $\alpha = \beta = 30^\circ$, we have $\cos(\alpha + \beta) = \cos 60^\circ = \frac{1}{2}$, which is not equal to $\cos\alpha + \cos\beta = 2\cos 30^\circ = \sqrt{3}$. Naturally, we might ask ourselves questions such as how $\sin\alpha$, $\sin\beta$, and $\sin(\alpha + \beta)$ relate to one another.

Consider the diagram shown on the right-hand side above. Let DEF be a right triangle with $\angle DEF = 90^\circ$, $\angle FDE = \beta$, and $DF = 1$ inscribed in the rectangle $ABCD$. (This can always be done in the following way. Construct line ℓ_1 passing through D outside of triangle DEF such that lines ℓ_1 and DE form an acute angle congruent to α. Construct line ℓ_2 passing through D and perpendicular to line ℓ_1. Then A is the foot of the perpendicular from E to line ℓ_1, and C the foot of the perpendicular from F to ℓ_2. Point B is the intersection of lines AE and CF.)

Express $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$ in terms of $\sin\alpha, \sin\beta, \cos\alpha, \cos\beta$.

Solution: We compute the lengths of the segments inside this rectangle. In triangle DEF, we have $DE = DF\cos\beta = \cos\beta$ and $EF = DF\sin\beta = \sin\beta$. In triangle ADE, $AD = DE\cos\alpha = \cos\alpha\cos\beta$ and $AE = DE\sin\alpha = \sin\alpha\cos\beta$. Because $\angle DEF = 90^\circ$, it follows that $\angle AED + \angle BEF = 90^\circ = \angle AED + \angle ADE$, and so $\angle BEF = \angle ADE = \alpha$. (Alternatively, one may observe that right triangles ADE and BEF are similar to each other.) In triangle BEF, we have $BE = EF\cos\alpha = \cos\alpha\sin\beta$ and $BF = EF\sin\alpha = \sin\alpha\sin\beta$. Since $AD \parallel BC$, $\angle DFC = \angle ADF = \alpha + \beta$. In right triangle CDF, $CD = DF\sin(\alpha + \beta) = \sin(\alpha + \beta)$ and $CF = DF\cos(\alpha + \beta) = \cos(\alpha + \beta)$.

From the above, we conclude that

$$\cos\alpha\cos\beta = AD = BC = BF + FC = \sin\alpha\sin\beta + \cos(\alpha+\beta),$$

implying that

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Similarly, we have

$$\sin(\alpha+\beta) = CD = AB = AE + EB = \sin\alpha\cos\beta + \cos\alpha\sin\beta;$$

that is,

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

■

By the definition of the tangent function, we obtain

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos}\alpha\cos\beta - \sin\alpha\sin\beta = \frac{\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}}{1 - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$$

We have thus proven the *addition formulas* for the sine, cosine, and tangent functions for angles in a restricted interval. In a similar way, we can develop an addition formula for the cotangent function. We leave it as an exercise.

By setting $\alpha = \beta$ in the addition formulas, we obtain the *double-angle formulas*

$$\sin 2\alpha = 2\sin\alpha\cos\alpha,\ \cos 2\alpha = \cos^2\alpha - \sin^2\alpha,\ \tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha},$$

where for abbreviation, we write $\sin(2\alpha)$ as $\sin 2\alpha$. Setting $\beta = 2\alpha$ in the addition formulas then gives us the *triple-angle formulas*. We encourage the reader to derive all the various forms of the double-angle and triple-angle formulas. These are very important items in ones problem-solving tool chest.

Example 3.5. How large an equilateral triangle can you fit inside a 2-by-2 square?

Note: Let $ABCD$ denote the square. Equilateral triangle AEF is a largest that can fit inside $ABCD$ if E and F lies on BC and CD, respectively. (Why?) The reader should think about this thoroughly. We provide an argument following the two solution.

First Solution: Note that right triangles ABE and ADF are congruent. Set $BE = x$. Then $AE^2 = 4 + x^2$, $CE = CF = 2 - x$, and $EF^2 = 2(2-x)^2$. From $AE = EF$, we have $4 + x^2 = 2(2-x)^2$. Solving the equation gives $x = 4 - 2\sqrt{3}$ and $AE = AF = EF = \sqrt{2}(2-x) = 2\sqrt{6} - 2\sqrt{2}$.

Second Solution: Since right triangles ABE and ADF are congruent, $\angle BAE = \angle DAF = 15^\circ$. In right triangle ABE,

$$AE = \frac{AB}{\cos 15^\circ} = \frac{2}{\cos(45^\circ - 30^\circ)} = \frac{2}{\cos 45^\circ\cos 30^\circ + \sin 45^\circ \sin 30^\circ} = \frac{2}{\frac{\sqrt{2}}{4}(\sqrt{3}+1)} = 2\sqrt{2}(\sqrt{3}-1).$$

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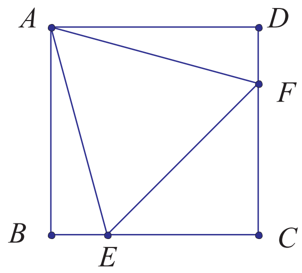
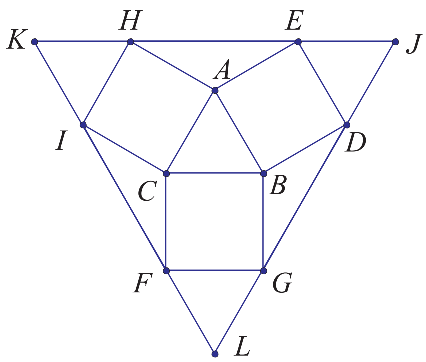




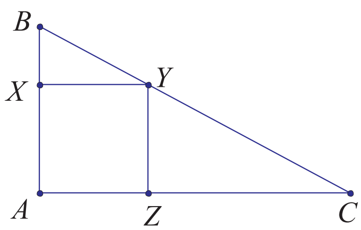
It is important to explain why triangle AEF is the largest. If equilateral triangle $\mathcal{T}$ is the largest can fit into the square, then all its vertices must lie on the sides of the square. (Otherwise, we can move it so it lies in the interior of the square and dilate it larger, contradicting the maximality assumption of $\mathcal{T}$.) If there is side of the square does not contain any of the vertices of $\mathcal{T}$, then can translate $\mathcal{T}$ towards this side, then at least one vertex of will not be on the sides of $ABCD$ any more, violating our previous observation. Thus, each side must contain at least one of the vertices of $\mathcal{T}$. Since the square has 4 sides and $\mathcal{T}$ has only 3 vertices, one of the vertices must be on two sides; that is, it is a vertex of the square. Without loss of generality, assume it is A, and we obtain the configuration discussed above.

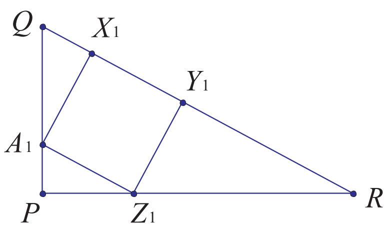
Example 3.6. [HMMT 2002] Equilateral triangle ABC of side length 2 is drawn. Three squares external to the triangle, $ABDE, BCFG$, and $CAHI$, are drawn. What is the area of the smallest triangle that covers these squares?

Solution: The desired triangle must covers the *convex hull* of the these squares; that is, it must cover hexagon $DEHIFG$. The equilateral triangle with sides lying on lines DG, EH, and FI has minimal area. (The only other reasonable candidate is the triangle with sides along DE, FG, HI, but a quick sketch shows that it is larger, since $HE > ED$.) Let J, K, and L be the vertices of this triangle closest to D, H, and F, respectively. Clearly, $KI = FL = 2$. Triangle FCI is a 30°-30°-120° triangle, so we can calculate the length of FI as $2\sqrt{3}$, making the side length of triangle JKL equal to $4 + 2\sqrt{3}$, and its area equal to $\frac{\sqrt{3}}{4}(4 + 2\sqrt{3})^2 = 12 + 7\sqrt{3}$. ■

Example 3.7. A square is *inscribed* in a right triangle if all of the vertices of the square lie on the sides of the triangle. Given a right triangle, there are two obvious ways to inscribe a square in it. The first is to place a corner of the square at the vertex with the right angle. The second way is to place a side of a square on the hypotenuse of the right triangle. Which way will result in a bigger square, or will they be equal?
Note: The first way is always better. We present two approaches.

First Proof:





We start with two congruent right triangles ABC and PQR. (We label them this way so we can compare these with the figure shown below for the second solution). Let $AB = PQ = c$, $BC = QR = a$, and $CA = RP = b$, and let $XY = s$ and $X_1Y_1 = t$. It suffices to show that $s^2 > t^2$.

Since triangles BXY and BAC are similar, we have

$$\frac{BY}{BC} = \frac{XY}{AC},$$

implying that $BY = \frac{sa}{b}$. Likewise, we also have $CY = \frac{sc}{a}$. Since $BY + YC = a$, we have $\frac{sa}{b} + \frac{sa}{c} = a$, implying that

$$s = \frac{bc}{b+c} \quad \text{or} \quad s^2 = \frac{b^2c^2}{(b+c)^2}.$$

Note that triangles A_1QX_1, RQP, and RZ_1Y_1 are similar. It follows that

$$QX_1 = \frac{ct}{b} \quad \text{and} \quad RY_1 = \frac{bt}{c}.$$

Because $QR = QX_1 + X_1Y_1 + Y_1R$ or $a = \frac{ct}{b} + t + \frac{bt}{c}$, it follows that

$$t = \frac{abc}{b^2+c^2+bc} \quad \text{or} \quad t^2 = \frac{a^2b^2c^2}{(b^2+c^2+bc)^2} = \frac{(b^2+c^2)b^2c^2}{(b^2+c^2+bc)^2}.$$

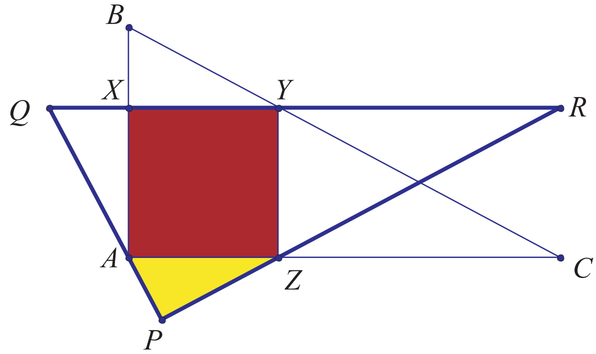
It remains to show that

$$\frac{(b^2+c^2)b^2c^2}{(b^2+c^2+bc)^2} = t^2 < s^2 = \frac{b^2c^2}{(b+c)^2}, \quad \text{or} \quad (b^2+c^2)(b+c)^2 < (b^2+c^2+bc)^2.$$

The inequality can easily be established by expanding its right-hand side as

$$\begin{aligned}(b^2+c^2+bc)^2 &= (b^2+c^2+bc)[(b+c)^2-bc] = (b^2+c^2)(b+c)^2 + [(b+c)^2-(b^2+c^2)]bc \\ &= (b^2+c^2)(b+c)^2 + 2b^2c^2.\end{aligned}$$

Second Proof: We inscribe the same square ($AXYZ$) into two similar right triangles. The second method requires a bigger triangle (triangle PQR), and so the first method (triangle ABC) is better. (Note that triangles YZR and YZC are congruent to each other, and so do triangles BXY and AXQ.)



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Note: Given a right triangle, how can you construct these two squares by compass and straight edge?

Exercises B

4.1. Show that a regular dodecagon can be cut into pieces that are all regular polygons, which need not all have the same number of sides.

4.2. Mark P inside square $ABCD$, so that triangle ABP is equilateral. Let Q be the intersection of BP with diagonal AC. Triangle CPQ looks isosceles. Is this actually true? If $AB = 1$, find the area of triangle APC.

4.3. Let $ABCD$ be a trapezoid with $AB \parallel CD$ and $CD = 2AB = 2AD$. Suppose that $BD = 6$ and $BC = 4$, find the area of the trapezoid.

4.4. Square $ABCD$ has sides of length 1. Points E and F lie on sides BC and CD, respectively, so that triangle AEF is equilateral. A square with vertex B has sides that are parallel to those of $ABCD$ and a vertex on segment AE. What is side length of this smaller square?

4.5. Given that one of the angles of the triangle with sides $(5, 7, 8)$ is $60°$, show that one of the angles of the triangle with sides $(3, 5, 7)$ is $120°$.

4.6. Show that one of the angles of the triangle with sides $(5, 7, 8)$ is $60°$.

4.7. Draw an 8-by-9-by-12 box $ABCDEFGH$. How many right triangles can be formed by connecting three of the eight vertices?

4.8. Equilateral triangle ABC of side length 2 is drawn. Three squares containing the triangle, $ABDE$, $BCFG$, and $CAHI$, are drawn. What is the area of the smallest triangle that covers these squares?

4.9. A circle of radius r is concentric with and outside of a regular hexagon of side length 2. The probability that three entire sides of the hexagon are visible from a randomly chosen point on the circle is $\frac{1}{2}$. What is r?

4.10. Let ABC be a right triangle with $\angle C = 90°$. Two squares S_1 and S_2 are inscribed in triangle ABC such that S_1 and ABC share a common vertex C, and S_2 has one of its sides on AB. Suppose that $[S_1] = 441$ and $[S_2] = 440$. Calculate $AC + BC$. Please solve this problem without Calculator assistance.

Special angles

Example 5.1. [AHSME 1999] The equiangular convex hexagon $ABCDEF$ has $AB = 1$, $BC = 4$, $CD = 2$, and $DE = 4$. Find $[ABCDEF]$.

Solution: Lines BC and DE meet at Q, lines DE and FA meet at R, and lines FA and BC meet at P. Then PQR, PAB, QCD, REF are equilateral triangles with $PQ = PB + BC + CQ = AB + BC + CD = 7$ and $RE = RQ - QD - DE = PQ - CD - DE = 1$. Thus,

$$[ABCDEF] = [PQR] - [PAB] - [QCD] - [REF] = \frac{\sqrt{3}}{4} \cdot (7^2 - 1^2 - 2^2 - 1^2) = \frac{43\sqrt{3}}{4}.$$

■

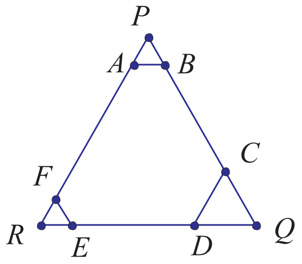
Example 5.2. Determine if one can find a equilateral hexagon with side length $\sqrt{29}$ and all its vertices being lattice points? If *yes*, give an example and find the area of the hexagon; if *no*, explain why.

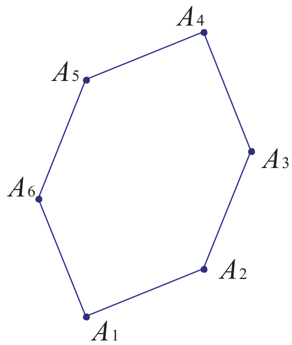
Solution: The answer is *positive*.

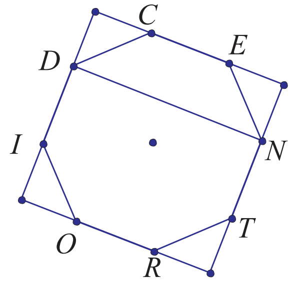
Note that $29 = 5^2 + 2^2$. We consider vectors $[\pm 5, \pm 2]$ and $[\pm 2, \pm 5]$. Let $A_1 = (0,0)$, $A_2 = (5,2)$, $A_3 = A_2 + [2,5] = (7,7)$, $A_4 = A_3 + [-2,5] = (5,12)$, $A_5 = A_4 + [-5,-2] = (0,10)$, and $A_6 = A_5 + [-2,-5] = (-2,5)$. Then $A_1A_2A_3A_4A_5A_6$ is equilateral with side length $\sqrt{29}$. We also have

$$[A_1A_2A_3A_4A_5A_6] = [A_1A_6A_5] + [A_1A_2A_4A_5] + [A_2A_3A_4] = 10 + 50 + 10 = 70.$$

■







Example 5.3. A stop sign — a regular *octagon* — can be formed from a 12-inch square sheet of metal by making four straight cuts that snip off the corners. How long are the sides of the resulting polygon?

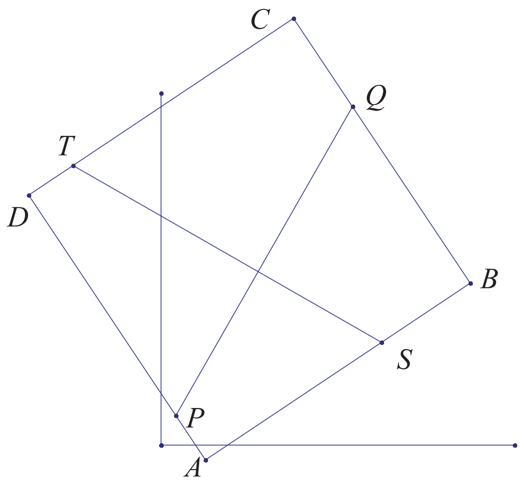
Solution: Let $CENTROID$ denote the stop sign. Set $CE = x$. Then

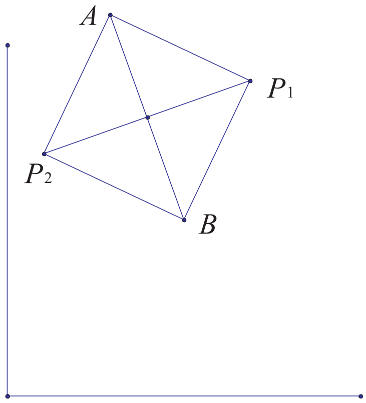
$$12 = DN = \frac{DC}{\sqrt{2}} + CE + \frac{EN}{\sqrt{2}} = x\sqrt{2} + x,$$

implying that $x = 12(\sqrt{2} - 1)$. ■

Example 5.4. Let $A = (3,-1)$, $B = (21,11)$, $C = (9,29)$, $D = (-9,17)$, $P = (1,2)$, and $Q = (13,23)$. A pair (S,T) of lattice points is *super* if S lies on segment AB and T on CD such that $ST = PQ$. Find two pairs of super points.

Solution: Since $\overrightarrow{AB} = \overrightarrow{DC} = [18,12]$ and $\overrightarrow{AD} = \overrightarrow{BC} = [-12,18]$, $ABCD$ is a square. Since $\overrightarrow{AP} = [-2,3] = \overrightarrow{AD}/6$ and $\overrightarrow{CQ} = [4,-6] = \overrightarrow{CB}/3$, P lies on AD and Q lies on BC. A 90° degree rotation can send P and Q to points on sides AB and CD, respectively. More precisely, since $\overrightarrow{PQ} = [12,21]$, we can set $\overrightarrow{ST} = [-21,12]$. We can then choose S to be any lattice point on segment AB to obtain a pair of super points S and T. (For example, $(S,T) = ((12,5),(-9,17))$ and $(S,T) = ((15,7),(-6,19))$.) ■





Example 5.5. Given $A(7, 26)$ and $B(12, 12)$, find a point P such that $AP = BP$ and $\angle APB = 90°$.

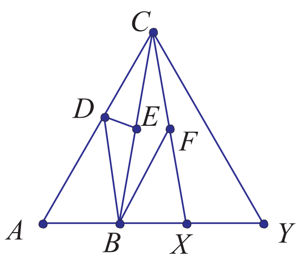
Solution: Note that triangle ABP is an isosceles right triangle with $AP = BP$. Let M be the midpoint of segment AB. Then $M = \left(\frac{19}{2}, 19\right)$, $MA = MB = MP$, and $MA \perp MB$. Thus $\overrightarrow{MA} = \left[\frac{5}{2}, -7\right]$, and $\overrightarrow{MP} = \left[7, \frac{5}{2}\right]$ or $\overrightarrow{MP} = -\left[7, \frac{5}{2}\right]$. Let $O = (0, 0)$ be the origin. It follows that $\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP} = \left[\frac{19}{2}, 19\right] \pm \left[7, \frac{5}{2}\right]$. Consequently, $P = \left(\frac{33}{2}, \frac{43}{2}\right)$ or $P = \left(\frac{5}{2}, \frac{33}{2}\right)$. ■

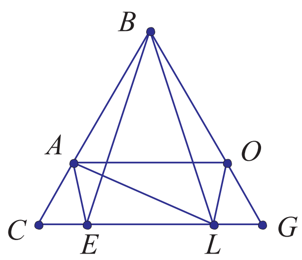
Example 5.6. In triangle ABC, $AC = 1$, $\angle BAC = 60°$, $\angle ABC = 100°$. Let E be the midpoint of segment BC, and let D be the point on segments AC such that $\angle DEC = 80°$. Evaluate $[ABC] + 2[CDE]$.

Solution: Extend segment AB through B to Y such that $ACY = 60°$. Then ACY is an equilateral triangle. Point X lies on segment AY such that $AB = XY$. Let F be the midpoint of CX. It is not difficult to see that CED and CFE, BED and BEF are two pairs of congruent triangles. Furthermore, since $CE = EB$, these four triangles have the same area. Therefore, $[ABC] + 2[CDE] = [ABC] + [BCF]$. By symmetry, $[ABC] = [CXY]$. Since $CF = FX$, $[BCF] = [BFX]$. Hence $[ACY] = [ABC] + [BCF] + [BXF] + [CXY] = 2([ABC] + [BCF])$, implying that

$$[ABC] + 2[CDE] = \frac{[ACY]}{2} = \frac{\sqrt{3}}{8}.$$

■





Example 5.7. In convex quadrilateral $ABLE$, $\angle ABE = 12°$, $\angle ALE = 24°$, $\angle EBL = 36°$, and $\angle BLA = 48°$. Compute $\angle AEL$.

First Solution: Set $\angle AEB = x$. Then $\angle LEB = 72° + x$. Triangle ALB is isosceles with $\angle ALB =$

$\angle ABL = 48^\circ$ and $AL = AB$. Applying the law of sines to triangles AEL and ABE gives

$$\frac{\sin 24^\circ}{\sin(72^\circ + x)} = \frac{AE}{AL} = \frac{AE}{AB} = \frac{\sin 12^\circ}{\sin x}.$$

We have $\sin 24^\circ \sin x = \sin 12^\circ \sin(72^\circ + x)$ or

$$2\cos 12^\circ \sin x = \sin(72^\circ + x) = \cos(x - 18^\circ), \qquad (*)$$

by the double-angle formula. It not difficult to see that $x = 30^\circ$ is a solution of $(*)$.

By the addition-subtraction formula, $(*)$ can be written as

$$2\cos 12^\circ \sin x = \cos x \cos 18^\circ + \sin x \sin 18^\circ \quad \text{or} \quad 2\cos 12^\circ - \sin 18^\circ = \cot x \cos 18^\circ.$$

The above equation has a unique solution for $\cot x$. Since $\cot x$ is one-to-one for $0 < x < 90^\circ$, $x = 30^\circ$ is the only solution for the problem and $\angle AEL = 102^\circ$.

Second Solution: Let C be the intersection of lines AB and EL. Then $\angle C = 60^\circ$. Extend segment EL through L to G such that $\angle LBG = 12^\circ$. Then BCG is an equilateral triangle. Point O lies on segment BG such that $BA = BO$. It is not difficult to see that, by symmetry, $AELO$ is an isosceles trapezoid with $AO \parallel EC$. Hence $\angle LAO = 24^\circ$. It is clear that ABO is an equilateral triangle. It is also clear that ABL is isosceles with $AB = AL$. It follows that AOL is an isosceles triangle with $\angle LAO = 24^\circ$ and $AL = AO$, and so $\angle ALO = 78^\circ$. Therefore, $\angle ELO = \angle ELA + \angle ALO = 102^\circ$, and by symmetry, $\angle AEL = 102^\circ$. ■

Exercises C

6.1. A triangle has a 60-degree angle and a 45-degree angle, and the side opposite the 45-degree angle is 240 units long. How long is the side opposite the 60-degree angle?

6.2. In equal angular octagon $ABCDEFGH$, $AB = CD = EF = GH = 6\sqrt{2}$ and $BC = DE = FG = HA$. Given the area of the octagon is 184, compute the length of side BC.

6.3. In the coordinate plane, what is the length of the shortest path from $(0, 0)$ to $(12, 16)$ that does not go inside the circle $(x - 6)^2 + (y - 8)^2 = 25$?

6.4. Let $ABCDE$ be a convex pentagon with $DC \parallel AB$ and $AE \perp ED$. Given that $\angle BAE = \angle BCD$ and $\angle D = 130^\circ$, compute $\angle B$.

6.5. In trapezoid $ABCD$ with $BC \parallel AD$, let $BC = 1000$ and $AD = 2008$. Let $\angle A = 37^\circ$ and $\angle B = 53^\circ$, and M and N be the midpoints of sides BC and AD, respectively. Find the length MN.

6.6. Let $ABCD$ be a parallelogram, and let M and N be the midpoints of sides BC and CD, respectively. Suppose that $AM = 2$, $AN = 1$, and $\angle MAN = 60^\circ$. Compute AB.

6.7. Let ABC be a triangle with $AB = AC$ and $\angle BAC = 20^\circ$, and let P be a point on side AB such that $AP = BC$. Find $\angle ACP$.

6.8. In convex quadrilateral $ABCD$, $AB = BC = CD$, $\angle B = 70^\circ$, and $\angle C = 170^\circ$. Find $\angle A$.

6.9. The points $(0,0), (a,11)$, and $(b,37)$ are the vertices of an equilateral triangle. Find a and b.

6.10. Consider the points $A = (0,12)$, $B = (10,9)$, $C = (8,0)$, and $D = (-4,7)$. There is a unique square $\mathcal{S}$ such that each of the four points is on a different side of $\mathcal{S}$. Let K be the area of $\mathcal{S}$. Find the remainder when $10K$ is divided by 1000.