Graph Paper and Vectors

1. A *lattice point* is a point whose coordinates are *integers*. Find two lattice points that are exactly $\sqrt{13}$ units apart. Is is possible to find lattice points that are $\sqrt{15}$ units apart? Is it possible to form a square whose area is 18 by connecting four lattice points? Explain.

2. Find an example of an equilateral hexagon whose sides are all $\sqrt{13}$ units long. Give coordinates for all six points.

3. A small red bug moves linearly with constant speed across my graph paper. I first notice the bug when it is at $(3, 4)$. It reaches $(9, 8)$ after two seconds and $(15, 12)$ after four seconds.
(a) Predict the position of the bug after six seconds; after nine seconds; after t seconds.
(b) Is there a time when the bug is equidistant from the x- and y-axes? If so, where is it?

4. The x- and y-coordinates of a point are given by the equations shown below. The position of the point depends on the value assigned to t. Use your graph paper to plot points corresponding to the values $t = -4$, -3, -2, -1, 0, 1, 2, 3, and 4. Do you recognize any patterns? Describe them.

$$\begin{cases} x = 2 + 2t \\ y = 5 - t \end{cases}$$

5. (Continuation) Plot the following points on the coordinate plane: $(1, 2)$, $(2, 5)$, $(3, 8)$. Write equations, similar to those in the preceding exercise, that produce these points when t-values are assigned. There is more than one correct answer.

6. In a dream, Blair is confined to a coordinate plane, moving along a line with a constant speed. Blair's position at 4 am is $(2, 5)$ and at 6 am it is $(6, 3)$. What is Blair's position at 8:15 am when the alarm goes off?

7. Draw the following segments. What do they have in common?
from $(3, -1)$ to $(10, 3)$; from $(1.3, 0.8)$ to $(8.3, 4.8)$; from $(\pi, \sqrt{2})$ to $(7 + \pi, 4 + \sqrt{2})$.

8. (Continuation) The *directed segments* have the *same* length and the *same* direction. Each represents the *vector* $[7, 4]$. The *components* of the vector are the numbers 7 and 4.
(a) Find another example of a directed segment that represents this vector. The initial point of your segment is called the *tail* of the vector, and the final point is called the *head*.
(b) Which of the following directed segments represents $[7, 4]$? from $(-2, -3)$ to $(5, -1)$; from $(-3, -2)$ to $(11, 6)$; from $(10, 5)$ to $(3, 1)$; from $(-7, -4)$ to $(0, 0)$.

9. A bug is moving along the line $3x + 4y = 12$ with constant speed 5 units per second. The bug crosses the x-axis when $t = 0$ seconds. It crosses the y-axis later. When? Where is the bug when $t = 2$? when $t = -1$? when $t = 1.5$? What does a negative t-value mean?

10. Give the components of the vector whose length is 10 and whose direction *opposes* the direction of $[-4, 3]$.

11. Find parametric equations to describe the line $3x+4y = 12$. Use your equations to find coordinates for the point that is three-fifths of the way from $(4, 0)$ to $(0, 3)$. By calculating some distances, verify that you have the correct point.

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1. A bug moves at 13 cm/sec, its position described by the parametric equations at right. Explain. Change the equations to obtain the description of a bug moving along the same line with speed 26 cm/second.

$$\begin{cases} x = 2 - 12t \\ y = 1 + 5t \end{cases}$$

2. Given the vector $[-5, 12]$, find the following vectors:
(a) same direction, twice as long **(b)** same direction, length 1
(c) opposite direction, length 10 **(d)** opposite direction, length c

3. Find the lengths of the following vectors:
(a) $[3, 4]$ **(b)** $1998\,[3, 4]$ **(c)** $\frac{1998}{5}\,[3, 4]$ **(d)** $t[3, 4]$ **(e)** $t[a, b]$

4. Find the number that is two thirds of the way **(a)** from -7 to 17; **(b)** from m to n.

5. The components of vector $[24, 7]$ are 24 and 7. Find the components of a vector that is three fifths as long as $[24, 7]$.

6. Let $A = (-5, 2)$ and $B = (19, 9)$. Find coordinates for the point P between A and B that is three fifths of the way from A to B. Find coordinates for the point Q between A and B that is three fifths of the way from B to A.

7. Find coordinates for the vertices of a *lattice rectangle* that is three times as long as it is wide, with none of the sides horizontal.

8. Motions of three particles are described by the following three pairs of equations

$$(a)\ \begin{cases} x = 2 - 2t \\ y = 5 + 7t \end{cases} \qquad (b)\ \begin{cases} x = 4 - 2t \\ y = -2 + 7t \end{cases} \qquad (c)\ \begin{cases} x = 2 - 2(t + 1) \\ y = 5 + 7(t + 1) \end{cases}$$

How do the positions of these particles compare at any given moment?

9. Robin is moving on the coordinate plane according to the rule $(x, y) = (-3+8t, 5+6t)$, where distance is measured in km and time is measured in hours. Casey is following 20 km behind, at the same speed. Write parametric equations describing Casey's motion.

10. Is it possible for a line to go through **(a)** *no* lattice points? **(b)** exactly *one* lattice point? **(c)** exactly *two* lattice points? For each answer, either give an example or else explain the impossibility.

11. Maintaining constant speed and direction for an hour, Whitney traveled from $(-2, 3)$ to $(10, 8)$. Where was Whitney after 35 minutes? What distance did Whitney cover in those 35 minutes?

12. Let $A = (0, 0)$ and $B = (12, 5)$, and let C be the point on segment AB that is 8 units from A. Find coordinates for C.

13. Let $A = (1, 4)$, $B = (8, 0)$, and $C = (7, 8)$. Find the area of triangle ABC.

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1. Show that two vectors $[a, b]$ and $[c, d]$ are perpendicular if, and only if, $ac + bd = 0$. The number $ac + bd$ is called the *dot product* of the vectors $[a, b]$ and $[c, d]$.

2. Find the area of the triangle whose vertices are $A = (-2, 3)$, $B = (6, 7)$, and $C = (0, 6)$.

3. Given points $A = (0, 0)$ and $B = (-2, 7)$, find coordinates for C and D so that $ABCD$ is a square.

4. Brett and Jordan are out driving in the coordinate plane, each on a separate straight road. The equations $B_t = (-3, 4) + t[1, 2]$ and $J_t = (6, 1) + t[-1, 1]$ describe their respective travels, where t is the number of minutes after noon.
(a) Make a sketch of the two roads, with arrows to indicate direction of travel.
(b) Where do the two roads intersect?
(c) How fast is Brett going? How fast is Jordan going?
(d) Do they collide? If not, who gets to the intersection first?

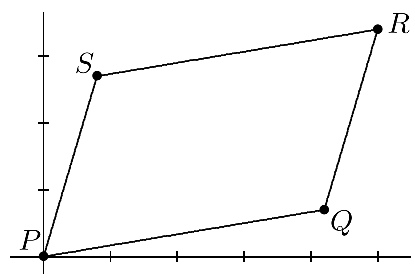
5. Find k so that the vectors $[4, -3]$ and $[k, -6]$
(a) point in the same direction; **(b)** are perpendicular.

6. An object moves with constant *velocity* (which means constant speed and direction) from $(-3, 1)$ to $(5, 7)$, taking five seconds for the trip.
(a) What is the speed of the object?
(b) Where does the object cross the y-axis?
(c) Where is the object three seconds after it leaves $(-3, 1)$?

7. Find the area of the parallelogram whose vertices are $(0, 0)$, $(7, 2)$, $(8, 5)$, and $(1, 3)$.

8. Find a vector that is perpendicular to the line $3x - 4y = 6$.

9. Given the points $A = (0, 0)$, $B = (7, 1)$, and $D = (3, 4)$, find coordinates for the point C that makes quadrilateral $ABCD$ a parallelogram. What if the question had requested $ABDC$ instead?



10. The figure at right shows a parallelogram $PQRS$, three of whose vertices are $P = (0, 0)$, $Q = (a, b)$, and $S = (c, d)$.
(a) Find the coordinates of R.
(b) Find the area of $PQRS$, and simplify your formula.

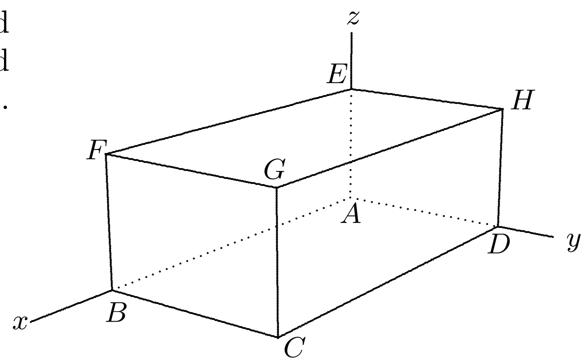
11. A stop sign — a regular *octagon* — can be formed from a 12-inch square sheet of metal by making four straight cuts that snip off the corners. How long, to the nearest 0.01 inch, are the sides of the resulting polygon?

12. How large an equilateral triangle can you fit inside a 2-by-2 square?

13. Find the area of the parallelogram whose vertices are $(2, 5)$, $(7, 6)$, $(10, 10)$, and $(5, 9)$.

Graph Paper and Vectors

1. The diagram shows a rectangular box named $ABCDEFGH$. Notice that $A = (0, 0, 0)$, and that B, D, and E are on the coordinate axes. Given that $G = (6, 3, 2)$, find
(a) coordinates for the other six vertices;
(b) the lengths AH, AC, AF, and AG;
(c) the distance from G to the z-axis.



2. While the Wood's Hole-Martha's Vineyard ferryboat steamed along at 8 mph through calm seas, passenger Dale exercised by walking the perimeter of the rectangular deck, at a steady 4 mph. Discuss the variations in Dale's actual speed, *relative to the water.*

3. Find coordinates for the point where line $(x, y) = (-1+2t, 3-t)$ meets line $y = 2x-5$.

4. Let $P = (-1, 3)$. Point Q has the property that all the points on the line $y = 2x - 5$ are equally distant from P and Q. Find coordinates for Q.

5. Draw a parallelogram whose adjacent edges are determined by vectors $[2, 5]$ and $[7, -1]$, placed so that they have a common initial point. This is called placing vectors *tail-to-tail.*
(a) Find the area of the parallelogram.
(b) Find a different pair of vectors that describes the same parallelogram.

6. Let $A = (5, -3, 6)$, $B = (0, 0, 0)$, and $C = (3, 7, 1)$. Show that ABC is a right angle.

7. Show that the vectors $[4, 5, -3]$ and $[2, -1, 1]$ are perpendicular.

8. Find components for a vector of length 21 that points in the same direction as $[2, 3, 6]$.

9. Find coordinates for the point on segment KL that is 5 units from K, where
(a) $K = (0, 0, 0)$ and $L = (4, 7, 4)$; **(b)** $K = (3, 2, 1)$ and $L = (7, 9, 5)$.

10. Write an equation that says that points $(0, 0, 0)$, (a, b, c), and (m, n, p) form a right triangle, the right angle being *at the origin.* Simplify your equation as much as you can.

11. Write an equation that says that vectors $[a, b, c]$ and $[m, n, p]$ are perpendicular.

12. Write an equation that says that vectors $[a, b]$ and $[m, n]$ are perpendicular.

13. Give an example of a vector perpendicular to $[6, 2, 3]$ that has the same length.

14. Find coordinates for a point that is 7 units from the line $4x + 3y = 12$.

15. The vectors $[8, 0]$ and $[3, 4]$ form a parallelogram. Find the lengths of its altitudes.

Graph Paper and Vectors

1. What is the radius of the largest circle that you can draw on graph paper that encloses
(a) *no* lattice points? **(b)** exactly *one* lattice point?
(c) exactly *two* lattice points? **(d)** exactly *three* lattice points?

2. There is a first-quadrant point on the line $y = 5x/12$ that is 8 units from the origin. Find its coordinates.

3. Given $A = (3, 1)$ and $B = (10, -3)$, find coordinates for two points C and D that make $ABCD$ a square.

4. An object moves at constant speed along the line $4x + 3y = 12$. It leaves the point $(0, 4)$ when $t = 0$, and reaches the point $(3, 0)$ when $t = 1$.
(a) Where is the object when $t = 2$?
(b) Write the parametric equations for this line.
(c) At what time t does the object cross the line $y = x$?

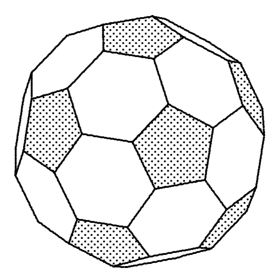
5. Find coordinates for a lattice-point rectangle whose sides are $\sqrt{29}$ and $2\sqrt{29}$ units long.

6. Write parametric equations for the line that goes through $(6, 7)$ and that crosses the line $4x + 3y = 12$ *perpendicularly.* Then find the x-intercept of the line traced by your parametric equations.

7. Write a formula, in terms of k and m, for the number that is exactly three fifths of the way from k to m.

8. Let $A = (-2, 7)$ and $B = (5, -3)$. Where does segment AB cross the x-axis?

9. Are the points (5,2,4), (9,14,10), and (11,20,13) collinear? Explain your answer.



10. A cube has 8 vertices, 12 edges, and 6 square faces. A soccer ball (also known as a *buckyball* or *truncated icosahedron*) has 12 pentagonal faces and 20 hexagonal faces. How many vertices and how many edges does a soccer ball have?

11. A rhombus has 25-cm sides, and one diagonal is 14 cm long. How long is the other diagonal?

12. [AHSME 1989] How many lattice points are on the segment whose endpoints are $(3, 17)$ and $(48, 281)$?
(A) 2 **(B)** 4 **(C)** 6 **(D)** 16 **(E)** 46

13. [AHSME 1998] A piece of graph paper is folded once so that $(0, 2)$ is matched with $(4, 0)$ and $(7, 3)$ is matched with (m, n). Find $m + n$.
(A) 6.7 **(B)** 6.8 **(C)** 6.9 **(D)** 7.0 **(E)** 8.0

Graph Paper and Vectors

1. It looks like the lattice points $(0,0)$, $(15,4)$, and $(4,15)$ form an equilateral triangle. Do they?

2. It is possible to find six lattice points that form an equilateral hexagon. Is it possible to find five lattice points that form an equilateral pentagon?

3. [AHSME 1993] Which of the following could *not* be the lengths of the external diagonals of a rectangular box?
(A) $\{4,5,6\}$ **(B)** $\{4,5,7\}$ **(C)** $\{4,6,7\}$ **(D)** $\{5,6,7\}$ **(E)** $\{5,7,8\}$