Invariants

Oleg Golberg

1 Numeric Invariants

Problem 1. Given three numbers, we can change any two of them, a, b to $\frac{a+b}{\sqrt{2}}$ and $\frac{a-b}{\sqrt{2}}$. Can we obtain $(1, \sqrt{2}, 1+\sqrt{2})$ from $(2, \sqrt{2}, \frac{1}{\sqrt{2}})$ with a series of such operations?

Problem 2. Let S be a set of numbers. If $a, b \in S$, we can add $ab + a + b$ to S. Initially $S = \{2, 3, 4, 5\}$. Is it possible that after a series of such operations $3023 \in S$?

Problem 3. There are three piles of stones. We can select two of them, take a stone from the first pile, write down the difference between the number of stones in the second pile and the number of stones in the first pile, and put the stone into the second pile. After a series of such operations each pile contains the same number of stones as it did initially. What is the sum of the number we have written down?

Problem 4. Real numbers are written around a circle. If three consecutive numbers a, b, c, d are such that $(a-d)(b-c) > 0$, then we can swap b and c. Prove that this operation can be performed only finitely many times.

2 Permutations and Parity

Definition 1. A permutation of size n is a bijection from $\{1, 2, \ldots, n\}$ to itself. The set of all permutations of size n is denoted by S_n.

Definition 2. An inversion in $\sigma \in S_n$ is a pair (i, j) such that $i < j$ but $\sigma(i) > \sigma(j)$.

Definition 3. The parity of $\sigma \in S_n$ us the parity of the number of inversions in σ.

Definition 4. The transposition of $i \neq j$ in $\sigma \in S_n$ is swapping the values of $\sigma(i)$ and $\sigma(j)$.

Property 1. The transposition of i and $i+1$ changes the parity of a permutation.

Property 2. Any transposition changes the parity of the permutation.

Problem 5. In a usual fifteen-puzzle the pieces with 14 and 15 are switched and all other pieces are in order.

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

Can the puzzle be solved?

3 Miscellaneous

Problem 6. Given are 100 coins in a row in the following order: heads, tails, heads, ..., tails. We can select several consecutive coins and flip all of them. Find the minimum number of operations necessary to turn all coins tails up.

Problem 7. A graph has n vertices and the degree of any vertex does not exceed 5. Prove that the vertices can be split into three groups such that the number of edges with vertices in the same group does not exceed $\frac{n}{2}$.

4 Geometric Invariants

Problem 8. A triangle is given on the plane. We can take any its vertex and reflect it across another vertex forming a new triangle. If the initial triangle is equilateral, prove that with a series of such operations we cannot obtain a triangle with a smaller perimeter.

Problem 9. A triangle is given on the plane. We can its vertices A, B, C in some order and change B to such point B' that B, C, B' are collinear and $AB' = AB$ unless $B' = C$. If the initial triangle has sidelengths $5, 6, 7$, prove that we cannot obtain a triangle of perimeter greater than 19 with a series of such operations.

Problem 10. Prove that for each $n \geq 3$ any n distinct points on the plane can be connected by pairwise nonintersecting segments that form polygon.

Problem 11. Let P be a convex $2n$-gon. Point X is inside P but does not lie on any of its diagonals. Prove that the number of triangles with vertices at the vertices of P that contain X is even.

Problem 12. Let $n \geq 2$ be a positive integer. Initially, there are n fleas on a horizontal line, not all at the same point. For a positive real number λ, define a move as follows:

- choose any two fleas, at points A and B, with A to the left of B;
- let the flea at A jump to the point C on the line to the right of B with $\frac{BC}{AB} = \lambda$.

Determine all values of λ such that, for any point M on the line and any initial positions of the n fleas, there is a finite sequence of moves that will take all the fleas to positions to the right of M.

Homework Problems

Problem 1. Initially there is a pile of 1001 stones. A valid move consits of removing a stone from a pile that contains at least two stones and splitting a pile (perhaps, another) into two nonempty piles. Can it occur that after a series of valid moves each pile contains exactly three stones.

Problem 2. Three hockey pucks lie on the ice rink. A player can hit one of the pucks so that it passes between the two other pucks. Prove that only after an even number of hits all pucks may return to their original positions.

Problem 3. To each vertex of a regular pentagon an integer is assigned, so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively, and $y < 0$, then the following operation is allowed: x, y, z are replaced by $x + y$, $-y$, $z + y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.