Bijection

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Introduction and examples

2.1.1. we introduce a fundamental technique in solving combinatorial problems.

> In order to count the elements of a certain set, we replace them with those of another set that has the same number of elements and whose elements are more easily counted.

Let A and B be finite sets, and let f be an injective function from A to B. Then there are at least as many elements in B as in A. Furthermore, if f is bijective, then A and B have the same number of elements.

2.1.2. [AMC10A/12A 2005] How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?

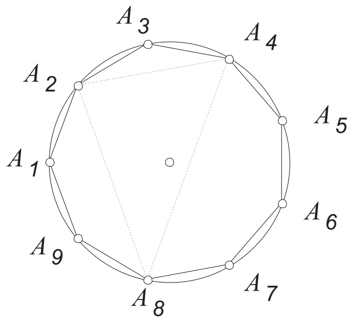
Comment: We can choose the following approach: For fixed middle digit, count the number of pairs of digits with the middle digit as their average. But this is not simple.

Solution: e note that the first and last digits must be both even or both odd for their average to be an integer. There are $5 \cdot 5 = 25$ odd-odd combinations for the first and last digits. There are $4\dot{5} = 20$ even-even combinations that do not use 0 as the first digit. Hence the total is 45.

2.1.3. Each of the vertices of a regular nonagon has been colored either red or blue. Prove that there exist two congruent monochromatic triangles; that is, triangles whose vertices are all the same color.

Proof: We call a monochromatic triangle red (blue) if all of its vertices are red (blue). Because there are nine vertices colored in two colors, at least five must be of the same color. Without loss of generality, we say that this color is red. Hence there are at least $\binom{5}{3} = 10$ red triangles. We now prove that there are two congruent red triangles.

Let $A_1, A_2, \cdots, A_9$ denote the vertices of the nonagon (Figure 4.1), and let ω be the its circumcircle. The vertices of the nonagon cut ω into nine equal arcs. We call each of these nine arcs a *piece*. Let $A_iA_jA_k$ be a triangle with $A_iA_j \le A_jA_k \le A_kA_i$. Denote by $a_{i,j}$ the number of pieces in the arc $\widehat{A_iA_j}$, not containing the point A_k, and define $a_{j,k}$ and $a_{k,i}$ analogously. Then define a map that maps the triangle $A_iA_jA_k$ to the triple $(a_{i,j}, a_{j,k}, a_{k,i})$. It is clear that $1 \le a_{i,j} \le a_{j,k} \le a_{k,i} \le 7$ and $a_{i,j} + a_{j,k} + a_{k,i} = 9$. For example, the triangle with vertices A_2, A_4, A_8 is read as triangle $A_4A_2A_8$ and mapped to the triple $(2, 3, 4)$.



Congruent triangles map to the same triple, while incongruent triangles map to distinct triples. Hence we have built a bijection between the classes of congruent triangles and the set of ordered triples of positive integers (a, b, c) with $a \le b \le c$ and $a+b+c = 9$. It is not difficult to list all such triples: $(1, 1, 7)$, $(1, 2, 6)$, $(1, 3, 5)$, $(1, 4, 4)$, $(2, 2, 5)$, $(2, 3, 4)$, $(3, 3, 3)$. Hence there are seven classes of congruent triangles. Since there are at least 10 red triangles, some class must contain at least two red triangles and hence there are at least two congruent red triangles.

2.1.4. Let n be a positive integer. In how many ways can one write a sum of (at least two) positive integers that add up to n? Consider the same set of integers written in a different order as being different. (For example, there are 3 ways to express 3 as $3 = 1 + 1 + 1 = 2 + 1 = 1 + 2$.)

Comment: We may view $3 = 2 + 1 = (1 + 1) + 1$.

Let m and n be positive integers. Show that

(a) There are $\binom{n-1}{m-1}$ ordered m-tuples $(x_1, x_2, \ldots, x_m)$ of positive integers satisfying the equation $x_1 + x_2 + \cdots + x_m = n$.

(b) There are $\binom{n+m-1}{m-1}$ ordered m-tuples $(x_1, x_2, \ldots, x_m)$ of nonnegative integers satisfying the equation $x_1 + x_2 + \cdots + x_m = n$.

Comment: This is similar to the previous problem.

2.1.5. [AIME 2000] Given eight distinguishable rings, find the number of possible five-ring arrangements on the four fingers (not the thumb) of one hand. (The order of the rings on each finger is significant, but it is not required that each finger have a ring.)

Comment: This is an quick application of the above.

2.1.6. [AHSME 1992] Ten points are selected on the positive x-axis, $\mathbf{X}^+$, and five points are selected on the positive y-axis, $\mathbf{Y}^+$. The fifty segments connecting the ten points on $\mathbf{X}^+$ to the five points on $\mathbf{Y}^+$ are drawn. What is the maximum possible number of points of intersection of these fifty segments in the interior of the first quadrant?

Comment: A point of intersection in the first quadrant is obtained whenever two of the segments cross to form an $\times$.

2.1.7. [China 1991, by Weichao Wu] Let n be an integer with $n \ge 2$, and define the sequence $S = (1, 2, \ldots, n)$. A subsequence of S is called arithmetic if it has at least two terms and it is an arithmetic progression. An arithmetic subsequence is called maximal if this progression cannot be lengthened by the inclusion of another element of S. Determine the number of maximal arithmetic subsequences.

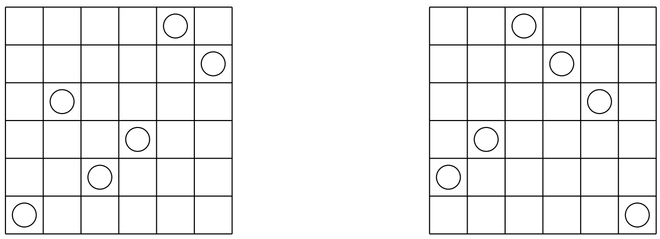
Comment: A maximal arithmetic sequence must have an element greater than (less than) or equal to $\frac{n}{2}$.

2.1.8. [PEA Math Materials] Suppose that two PEA administrators are the only persons who have dial-in access to the Academy's Internet fileserver, which can handle only one call at a time. Each has a 15-minute project to do and hopes to use the fileserver between 4 PM and 6 PM today. Neither will call after 5:45 PM, and neither will call more than once. At least one of them will succeed. What is the probability that they both complete their tasks?

Comment: This is indeed an geometric probability problem. View a pair of arrival time as the coordinates of a point in the coordinate-plane.

2.1.9. [IMO 2002 Short–listed] Let n be a positive integer. A sequence of n (not necessarily distinct) positive integers is called *full* if it satisfies the following conditions: For each positive integer $k \geq 2$, if the number k appears in the sequence, then so does the number $k-1$; moreover, the first occurrence of $k-1$ comes before the last occurrence of k. For each n, how many full sequences are there?

Comment: Various solutions are possible for the above problem. Most of them use bijections. We provide a hint for one of the better bijections known. It is not difficult to see that there are $n!$ ways to place n non-attacking rooks on an $n \times n$ chessboard. For each of such placements, we can find a unique full sequence corresponding to it. For example, for $n = 6$, we can map the following placements



to sequences $(3, 2, 3, 3, 1, 2)$ and $(3, 3, 1, 2, 3, 4)$, respectively. Can you define this mapping and prove it is indeed a bijection?

Practice problems

2.1.10. Determine the number of subsets S of $\{1, 2, \ldots, 10\}$ with the following property: there exist integers $a < b < c$ with $a \in S$, $b \notin S$, and $c \in S$.

2.1.11. A certain lottery has tickets labeled with numbers $1, 2, \ldots, 1000$. The lottery is run as follows: First, a ticket is drawn at random. If the number on the ticket is odd, the drawing ends; if it is even, another ticket is randomly drawn (without replacement). If this new ticket has an odd number, the drawing ends; if it is even, another ticket is drawn (again without replacement), and so forth, until an odd number is drawn. Then every person ticket number was drawn (t any point of the process) wins a prize.

You have ticket number 1000. What is the probability that you get a prize?

2.1.12. The Security Bank of the Republic of Fatland has 15 senior executive officers. Each officer has an access card to the bank's vault. There are m distinct codes stored in the magnetic strip of each access card. To open the vault, each officer who is present puts his access card in the vault's electronic lock. The computer system then collects all of the distinct codes on the cards, and the vault is unlocked if and only if the set of the codes matches the set of n (distinct) preassigned codes. For security reasons, the bank's vault can be opened if and only if at least six of the senior officers are present. Find the values of n and m such that n is minimal and the vault's security policy can be achieved. (The elements in a set have no order.)

2.1.13. [AIME 2001, by Richard Parris] The numbers 1, 2, 3, 4, 5, 6, 7, and 8 are randomly written on the faces of a regular octahedron so that each face contains a different number. Compute the probability that no two consecutive numbers, where 8 and 1 are considered to be consecutive, are written on faces that share an edge.

Additional problems

2.1.14. [ARML 2001] Compute the number of sets of three distinct elements that can be chosen from the set

$$\left\{2^1, 2^2, 2^3, \dots, 2^{2000}\right\}$$

such that the three elements form an increasing geometric progression.

2.1.15. [AIME 2001] A fair die is rolled four times. What is the probability that each of the final three rolls is at least as large as the roll preceding it?

2.1.16. Let n be a positive integer. Points $A_1, A_2, \dots, A_n$ lie on a circle. For $1 \le i < j \le n$, we construct $\overline{A_iA_j}$. Let S denote the set of all such segments. Determine the maximum number of intersection points can produced by the elements in S.

2.1.17. Let $A = \{a_1, a_2, \dots, a_{100}\}$ and $B = \{1, 2, \dots, 50\}$. Determine the number of surjective functions f from A to B such that $f(a_1) \le f(a_2) \le \cdots \le f(a_{100})$. What if f does not need to be surjective?

2.1.18. [AIME 1983] For $\{1, 2, \dots, n\}$ and each of its nonempty subsets a unique *alternating sum* is defined as follows: Arrange the numbers in the subset in decreasing order and then, beginning with the largest, alternately add and subtract successive numbers. (For example, the alternating sum for $\{1, 2, 4, 6, 9\}$ is $9-6+4-2+1 = 6$ and for $\{5\}$, it is simply 5.) Find the sum of all such alternating sums for $n = 7$.

2.1.19. [St. Petersburg 1989] Tram tickets have six-digit numbers (from 000000 to 999999). A ticket is called *lucky* if the sum of its first three digits is equal to that of its last three digits. A ticket is called *medium* if the sum of all of its digits is 27. Let A and B denote the numbers of lucky tickets and medium tickets, respectively. Find $A - B$.

2.1.20. [Putnam 2002] Let n be an integer greater than one, and let T_n be the number of nonempty subsets S of $\{1, 2, 3, \dots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.