Pigeonhole

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What? Are those pigeons? And why are you stuffing them into holes?

Discussion

The *pigeonhole principle* states that, given positive integers m and n with $m > n$, to stuff m pigeons into n hole requires that at least two pigeons are stuffed into the same hole. More specifically, some hole contains at least $\lceil m/n \rceil$ pigeons. Furthermore, some hole contains at most $\lfloor m/n \rfloor$ pigeons. Simple and intuitive, this concept arises frequently in Olympiad problems.

Many characteristics can be assigned to the pigeons and holes; what is appropriate depends on the problem at hand. Some ideas include ordering within holes, forcing distinctness, and operations on pigeons. Even infinity plays an interesting, if infrequent, role. If an infinite set is partitioned into finitely many disjoint sets, then one of the subsets is itself infinite. In the opposite sense, a finite set cannot be partitioned into infinitely many disjoint nonempty subsets.

The problems below have quick solutions involving pigeonhole. What aspects of each tell us this? Questions of *existence* suggest an *averaging* approach that considers many things at once only to extract a small bit of abstract information. Residues modulo n (likely holes) are relatively few in number compared to subsets of an n element set (likely pigeons), and choosing slightly more than half of a collection makes that extra item extremely interesting. It is worth observing that the fact that subsets are so numerous makes them good pigeons. Think about the following problems before reading their solutions:

1. A set S contains n integers. Prove that there exists a nonempty subset $T \subset S$ such that the sum of the elements of T is a multiple of n.

2. (RMO, India 1998) Let n be a positive integer. Show that if S is a subset of $\{1, 2, 3, \ldots, 2n\}$ containing $n + 1$ elements, then (a) there are two distinct coprime elements of S; (b) there are two distinct elements of S such that one divides the other.

3. Suppose that in a room of $n \geq 2$ people, some pairs of people shake hands. Show that two people shook hands with the same number of other people.

4. (Zuming) The squares of an 8×8 checkerboard are filled with the numbers $\{1, 2, \ldots, 64\}$. Prove that some two adjacent squares (sharing a side) contain numbers differing by at least 5.

In the first problem, we can probably assume that residues modulo n will be the pigeons. With this in mind, we should need only about n pigeons. Given two subsets whose elements give equivalent remainders, we might adjust one set based on the other - we could use set subtraction if one set contained the other. Additionally, since the size of the subset T is not specified, it makes sense to consider the subsets of S as *posets* under inclusion. That is, set inclusion gives a partial ordering relation - an ordering where elements are not necessarily comparable - on the subsets. A natural inclusion structure is *ascending chains,* sequences of sets where one contains the next. So let the elements of S be $s_1, \ldots, s_n$, and define the subsets $T_k = \{s_1, s_2, \ldots, s_k\}$ for $k = 1, 2, \ldots, n$.

The solution is now clear. The partial sum corresponding to T_k is then $t_k = s_1 + \cdots + s_k$ for $1 \leq k \leq n$. If t_k is a multiple of n for some k, we are done. Otherwise, $t_i \equiv t_j \pmod{n}$ for some $i < j$, and in this case n divides $t_j - t_i = s_{i+1} + s_{i+2} + \cdots + s_j$.

The second problem clearly calls for two proofs based on pigeonhole, where $n + 1$ items are classified into n groups; pigeonhole then gives us a pair of items that grouped together. We must choose holes appropriately.

For (a), the holes should be pairs of relatively prime numbers. Although there probably are many possibilities, the Euclidean algortihm quickly verifies that consecutive integers are relatively prime, so we can use $\{1, 2\}, \{3, 4\}, \ldots, \{2n - 1, 2n\}$.

For (b), we need n equivalence classes such that for any two equivalent elements, one divides the other. If the numbers $m_1, m_2, \ldots, m_k$, are equivalent, then without loss of generality we may assume that m_1 divides m_2, that m_2 divides m_3, and so on. It makes sense to generate $m_2, \ldots, m_k$ from m_1 by adding prime factors, so as to pack as many numbers from $\{1, 2, 3, \ldots, 2n\}$ together. A natural candidate is the smallest prime, 2. Having reached this idea, it is actually easier to work in reverse, removing powers of 2.

Starting from an integer i such that $n < i \leq 2n$, remove the powers of 2 from i one at a time. The sequences of integers thus determined partition the set S into n equivalence classes based on common odd parts. We finally note that if x and y are two integers such that $x < y$ while both have the same odd part, then x divides y, and we are done.

In the third problem, the presumable holes are "number of other people whose hand was shook." Since there are n possibilities and n people, there is exactly one, very uniform fail case. So we suppose for the sake of contradiction that no two people shook hands with the same number of others. Then the people shook hands with $0, 1, 2, \ldots, n - 1$ others, one person per number. But this means that one person shook hands with nobody while another shook hands with everybody! Impossible.

To approach the last problem, we might begin by noting that the largest possible difference is $64 - 1 = 63$, much larger than we require. Indeed, as long as it is split into at most 15 differences, the maximal difference will give a solution via pigeonhole. It may happen that 1 and 64 are not adjacent. However, there we can clearly find a path between the squares labeled 1 and 64 using not more than 15 squares total. The 14 (or fewer) differences between the pairs of numbers in adjacent squares in the path sum to 63, so that, by pigeonhole, at least one such difference is at least 5.

Problems

1. Five lattice points are chosen in the plane lattice. Prove that two of these points determine a line segment passing through a third lattice point.

   Outline. Look at the parities of the coordinates of each point, then check that one of the midpoints is a lattice point.

2. Let set S contain five integers greater each than 1 and less than 120. Show that S contains a prime or two elements of S share a prime divisor.

   Outline. Suppose that the set $S = \{s_1, s_2, s_3, s_4, s_5\}$ contains no primes and that no two elements of S share a prime divisor. Then $s_i = p_{i1}p_{i2}$ for $i = 1, \dots, 5$. There are four primes less than 11. Thus, there exists an j such that $p_{j1}, p_{j2} \geq 11$. It follows that $s_j \geq 121 > 120$, a contradiction.

3. (Zuming) Let n be a positive integer that is not divisible by 2 or 5. Prove that there is a multiple of n consisting entirely of ones.

   Outline. Find two integers consisting entirely of 1's that are equivalent mod n, then remove the zeros from their difference.

4. Show that for any positive real ϵ there exist positive integers m and n such that $|m\pi - n| < \epsilon$.

   Outline. Fix a positive integer N such that $\frac{1}{N} < \epsilon$. The fractional part of any positive real lies in precisely one of the sets $[0, 1/N), [1/N, 2/N), \dots, [(N-1)/N, 1)$. Some two of elements of $\{\pi, 2\pi, \dots, (N+1)\pi\}$, have a fractional part in the same interval; their difference is what we need.

5. Show that any set of 27 different positive odd integers, all less than 100, contains a pair whose sum is 102.

   Solution. There are 24 pairs summing to 102:

   $$\{3, 99\}, \{5, 97\}, \dots, \{49, 53\}.$$

   The above pairs exclude 1 and 51, but choosing 25 of the above numbers requires choosing both numbers from at least one pair. □

6. Let n be a positive integer and $\mathcal{P}$ a set of n primes. Show that given any set S of $n+1$ positive integers whose prime divisors are contained in $\mathcal{P}$, we can find a subset $T \subset S$ such that the product of the elements of T is a perfect square.

   Outline. Consider the products of the elements of each subset. Use pigeonhole to find two subsets A and B that such that the products of the elements in each agree as far as the parities of the exponents of the primes in $\mathcal{P}$. Then use the set $(A \cup B) \setminus (A \cap B)$.

7. Does there exist an infinite sequence of positive integers, containing each positive integer exactly once, such that the sum of the first n terms is divisible by n for every n?

   Outline. Yes. Construct the sequence one element at a time, alternating between a large number and the least unused number. The large numbers are chosen so that their successors will satisfy the division property. They can be found with the help of the Chinese Remainder Theorem after noting that two consecutive integers are always relatively prime.

8. Let $n > 1$ be a positive integer. Suppose that $2n$ chess pieces are placed at distinct squares of an $n \times n$ chessboard. Show that four of these pieces determine the vertices of a parallelogram.

   Outline. Imagine each pair of pieces as determining a vector in the first or fourth quadrant. There are $n(2n-1)-1$ possibilities, owing to the board dimensions, but there are $n(2n-1)$ pairs.

9. (Zuming) Prove that every convex polygon with an even number of sides has a diagonal that is not parallel to any of its sides.

   Outline. For a $2n$-gon, there are $n(2n-3)$ diagonals and $2n$ sides, while each side is parallel to at most $n-2$ diagonals.

10. (Zuming) Let $a_1, \ldots, a_{100}$ and $b_1, \ldots, b_{100}$ be two permutations of the integers from 1 to 100. Prove that, among the products $a_1b_1, a_2b_2, \ldots, a_{100}b_{100}$, there are two with the same remainder upon division by 100.

    Outline. Suppose the remainders are all different. Then every residue appears once. Since there are 50 odd numbers, each pair $\{a_i, b_i\}$ comprises numbers of the same parity. It follows that every even number is a product of two even numbers, so a multiple of 4. Then there can be no remainder of the form $4k+2$.

11. The set $\{1, 2, 3, \ldots, 16\}$ is partitioned into three sets. Prove that one of the subsets contains some numbers x, y, z (not necessarily distinct) such that $x + y = z$.

    Outline. By pigeonhole, a subset, say A, contains at least six elements, $a_1 < a_2 < a_3 < a_4 < a_5 < a_6$. The differences $a_i - a_1$ as $i = 2, 3, \ldots, 6$ must lie in the other two sets, say B or C. Then WLOG B contains three of the differences, $b_1 < b_2 < b_3$. Then $b_2 - b_1, b_3 - b_1, b_3 - b_2$ lie in C yet $b_3 - b_1 = (b_3 - b_2) + (b_2 - b_1.)$

12. A 6×6 rectangular grid is tiled with nonoverlapping 1×2 rectangles. Must there be a line through the interior of the grid that does not pass through a rectangle?

    Outline. Yes. Using an inductive parity argument, we can prove that any of the 10 lines must pass through an even number of 1×2 rectangles. Then because there are just 18 small rectangles, one of the 10 lines pierces none.

Homework

1. Every edge and every diagonal of a regular hexagon is assigned one of two colors. Show that some three of these segments consititute a monochromatic triangle.

   Solution. Suppose the colors are red and blue. Label the hexagon $A_1A_2A_3A_4A_5A_6$. By the pigeon hole principle, some three edges from the set $\{\overline{A_1A_i} \mid i = 2, 3, \ldots, 6\}$ are the same color. Without loss of generality, $\overline{A_1A_2}$, $\overline{A_1A_3}$, and $\overline{A_1A_4}$ are blue. Then either the edges of $A_2A_3A_4$ are all red, or a blue edge in this triangle completes a blue triangle with vertex A_1. $\square$

2. Let $a_1, a_2, \ldots, a_{99}$ be a permutation of $1, 2, 3, \ldots, 99$. Prove that there exist two equal numbers from
$$|a_1 - 1|, |a_2 - 2|, \ldots, |a_{99} - 99|.$$

   Solution. Note that
$$\sum_{i=1}^{99} |a_i - i| \equiv \sum_{i=1}^{99} a_i - i = 0 \pmod 2. \quad (*)$$
   On the other hand, $|a_i - i| < 99$. Thus, if no two such numbers are equal, then
$$\sum_{i=1}^{99} |a_i - i| = 0 + 1 + \cdots + 98 = 49 \cdot 99 \equiv 1 \pmod 2,$$
   contrary to (*). $\square$

3. (RMO, India 1996) Suppose that A is a 50-element subset of $\{1, 2, 3, \ldots, 100\}$ such that no two numbers from A add up to 100. Show that A contains a square.

   Solution. Suppose for the sake of contradiction that A does not contain a square. Consider the pairs $\{1, 99\}, \{2, 98\}, \ldots, \{49, 51\}$. Observe that the intersection $A \cap \{36, 64\}$ is empty. It follows that regardless of whether $50 \in A$, the set A contains at least 49 elements from the 48 other pairs. By the pigeon hole principle, this contradicts the condition that A contains no pair of numbers adding up to 100. $\square$

4. (St. Petersburg, 1997) Fifty numbers are chosen from the set $\{1, \ldots, 99\}$, no two of which sum to 99 or 100. Prove that the chosen numbers are $50, 51, \ldots, 99$.

   Outline. Observe that no two of the chosen numbers can occur consecutively in the sequence $50, 49, 51, 48, 52, \ldots, 1, 99$.

5. (NMO, India 1994) Let S be a set of 181 square integers. Prove that there exists a 19-element subset $T \subset S$ such that the sum of the elements of T is divisble by 19.

   Outline. Note that there are just 10 quadratic residues in modulo 19; we can find 19 squares having the same remainder upon division by 19.

6. (Iran 1996) Suppose that a chessboard is made into a torus by identifying both pairs of opposite edges together. How many knights can be added to the resulting board such that no two knights attack each other?

   Outline. 32. It is possible to place the knights on all of the squares of one color. On the other hand, k knights occupy k squares and attack $8k$ squares, yet each of $64 - k$ squares can be attacked by at most 8 knights.

7. (Zuming) In each of 3 classes there are n students. Every student has exactly $n + 1$ friends altogether in the other two classes (not his class.) Prove that in each class you can find one student, so that all three of them, taken in pairs, are friends.

   Outline. Look at a student having the least number of friends in a different class; suppose the student is in class A, has k friends in class B, and at least $n-k+1$ friends in class C. Any friend of the student in class B must be friends with at least k students in class C. Then observe that the original student is unfamiliar with at most $k - 1$ students in class C.

8. (IMO 1972) Prove that from a set of ten distinct two-digit natural numbers, it is possible to select two disjoint nonempty subsets whose members have the same sum.

   Outline. There are $2^{10} = 1024$ subsets of a ten element set, but the sum of the numbers in any subset is a nonnegative integer less than 1000. Given two sets A and B whose elements have the same sum, we can find sets $A \setminus (A \cap B)$ and $B \setminus (A \cap B)$.