Counting

The Fundamental Counting Principle

In a compound event in which the first event may occur in n_1 different ways, the second event may occur in n_2 different ways and so on, and the k-th event may occur in n_k different ways, the total number of ways the compound event may occur is $n_1 n_2 n_3 \ldots n_k$.

Example 1: How many positive divisors does 2,250 have?

Definition: A *permutation* of a set of n objects is an ordered arrangement of the objects.

Example 2: Consider a set of 3 objects, {A, B, C, D}. How many ways are there to arrange these objects?

Theorem 1: The total number of permutation of a set of n is given by ${}_nP_n = n(n-1)(n-2) \ldots 3 \cdot 2 \cdot 1 = n!$. (n factorial)

Example 3 In how many ways can letter of the set {A, B, C, D, E, F, G} be arranged to form ordered codes of (a) 7 letters? (b) 5 letters? (c) 2 letters?

Theorem 2: The number of permutations of a set of n objects taken r ($r \le n$) at a time, denoted by ${}_nP_r$, is given by ${}_nP_r = n(n-1)(n-2) \ldots (n-r+1) = \dfrac{n!}{(n-r)!}$.

Example 4: In how many ways can the letters of the set {A, B, C, D, E} be arranged to form ordered codes of (a) 3 letters? (b) 7 letters if we allow repeated use of the same letter?

Theorem 3 The number of orderings of n objects taken r at a time, with repetition, is n^r.

Permutations of a set are arrangements of the elements of the set. Often we are concerned only with the number of ways we can select elements from a set. These are *combinations*.

Example 5: How many permutations and combinations are there of the set {A, B, C, D, E} taken 3 at a time?

Think of the process of finding the number of permutations of n objects taken r at a time in two steps: (1) select the r objects; (2) arrange the r objects.

Theorem 4: The number of combinations of a set of n objects taken r at a time, denoted by ${}_nC_r$ or by $\binom{n}{r}$, is given by $\binom{n}{r} = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r} = \dfrac{n!}{r!(n-r)!}$.

Problems:

1 How many four-digit positive integers have at least one digit that is a 2 or 3?

2 Bicycle license plates in Flatville each contain three letters. The first is chosen from the set {C, H, L, P, R}, the second from {A, I, O}, and the third from {D, M, N, T}. When Flatville needed more license plates, they added two new letters. The new letters may both be added to one set or one letter may be added to one set and one to another set. What is the largest possible number of ADDITIONAL license plates than can be made by adding two letters?

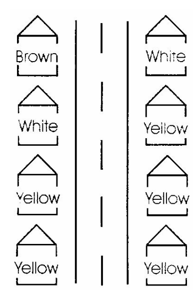
3 Using the letters A, M, O, S, and U, we can form 120 five-letter "words". If these "words" are arranged in alphabetical order, then what position does the "word" USAMO occupy?

4 Pat has plenty of 0's, 1's, 3's, 4's, 5's, 6's, 7's, 8's, and 9's, but he has only twenty-two 2's. How far can he number the pages of his scrapbook with these digits?

5 Three fair, standard six-faced dice of different colors are rolled. In how many ways can the dice be rolled such that the sum of the numbers rolled is 10?

6 A license plate consists of two letters followed by two digits, for example, MP78. Neither the digits nor the letters may be repeated, and neither the letter "O" nor the digit "0" may be used. When reading from the left to right, the letters must be in alphabetical order and the digits must be in increasing order. How many different license plate combinations are possible?

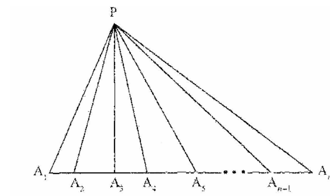
7 Each of eight houses on a street is painted brown, yellow or white. Each house is painted only one color and each color is used on at least one house. No two colors are used to paint the same number of houses. In how many ways could the eight houses on the street be painted? One such way is shown here.



8 How many different combinations of nickels, dimes and/or quarters equal exactly 60 cents?

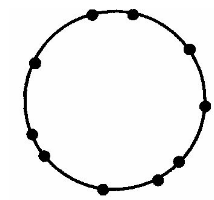
9 After a gymnastics meet, each gymnast shook hands with every other gymnast on every team. Afterwards, a coach came down and only shook hands with each gymnast for her own team. There were a total of 281 handshakes. What is the fewest number of handshakes the coach could have participated in?

10 The figure below presents a figure with a total of 120 triangles and n points labeled as vertices on the horizontal base. What is the value of n?

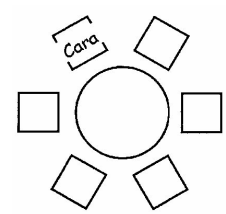


Homework:

1 Ten distinct points are identified on the circumference of a circle. How many different convex quadrilaterals can be formed if each vertex must be one of these 10 points?



2 Cara is sitting at a circular table with her five friends as shown to the right. How many different possible pairs of people could Cara be sitting between?



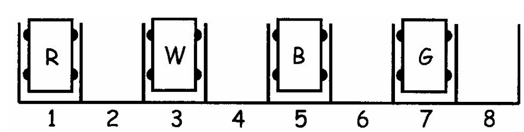
3 Four horizontal lines and four vertical lines are drawn in a plane. How many ways can four lines be chosen such that a rectangular region is enclosed?

4 How many non-empty subsets of {1, 2, 3, 4, 5, 6, 7, 8} consist entirely of odd numbers?

5 The positive four-digit integers that use each of the four digits 1, 2, 3, and 4 exactly once are ordered from least to greatest. What is the 15^{th} integer in the list?

6 How many times does the digit 6 appear in the list of all integers from 1 to 100?

7 A parking lot has a row of eight parking spaces numbered sequentially 1 through 8. Four cars (red, white, blue, and green) are parked such that no two cars are in adjacent parking spots. One such arrangement is shown here. How many arrangements are possible?



8 How many combinations of two or more consecutive positive integers have a sum of 45?