AM-GM inequality

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Introduction and examples

4.1.1. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} \le \frac{a}{c}+\frac{c}{b}+\frac{b}{a}.$$

4.1.2. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab} \ge a+b+c,$$

and determine when equality holds.

4.1.3. Determine the maximum value of constant λ such that

$$u+v+w \ge \lambda$$

u, v, and w be positive real numbers with $u\sqrt{vw}+v\sqrt{wu}+w\sqrt{uv} \ge 1$.

4.1.4. The dimensions of rectangular piece of paper $ABCD$ are $AB = 10$ and $BC = 9$. It is folded so that corner D is matched with a point F on edge BC. Let E be the point on side CD lying on the (folding) crease. Find the maximum value of the area of triangle EFC.

4.1.5. Let x and y be positive real numbers with $x^3 + y^3 = x - y$. Prove that $x^2 + 4y^2 < 1$.

4.1.6. Let x, y, z be nonnegative real numbers with $x + y + z = 1$. Prove that $xy + yz + zx - 9xyz \ge 0$.

4.1.7. For positive a, b, c prove that

$$2\sqrt{bc+ca+ab} \le \sqrt{3}\sqrt[3]{(b+c)(c+a)(a+b)}.$$

4.1.8. For positive real numbers x and y with $xy(x - y) = 8$, determine the minimum value of $x^2 + y^2$.

4.1.9. [Mathematics Magazine, Murray S. Klamkin] For positive real numbers a, b, c, d, prove that

$$a^4b + b^4c + c^4d + d^4a \ge abcd(a + b + c + d).$$

4.1.10. Let r, s, t be positive real numbers. Determine the minimum value of

$$(1+rst)\left[\frac{1}{r(1+s)}+\frac{1}{s(1+t)}+\frac{1}{t(1+r)}\right].$$

4.1.11. [IMO 2004 Shortlist] Let a, b, and c be positive real number with $ab + bc + ca = 1$. Prove that

$$\sqrt[3]{\frac{1}{a}+6b}+\sqrt[3]{\frac{1}{b}+6c}+\sqrt[3]{\frac{1}{c}+6a} \le \frac{1}{abc}.$$

4.1.12. Let a, b, c be nonnegative real numbers. Determine the minimum value of

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}.$$

4.1.13. Let $x_1, \ldots, x_n$ $(n \geq 2)$ be positive numbers satisfying

$$\frac{1}{x_1 + 1998} + \frac{1}{x_2 + 1998} + \cdots + \frac{1}{x_n + 1998} = \frac{1}{1998}.$$

Prove that

$$\frac{\sqrt[n]{x_1 x_2 \cdots x_n}}{n-1} \geq 1998.$$

4.1.14. Prove that

$$\sqrt[3]{\left(\frac{a}{b+c}\right)^2} + \sqrt[3]{\left(\frac{b}{c+a}\right)^2} + \sqrt[3]{\left(\frac{c}{a+b}\right)^2} \geq \frac{3}{\sqrt[3]{4}}.$$

4.1.15. Let x, y, z be nonnegative real numbers such that $x^2 + y^2 + z^2 = 1$. Prove that

$$\frac{1}{1-x} + \frac{1}{1-y} + \frac{1}{1-z} \geq \frac{9 + 3\sqrt{3}}{2}.$$