Solving Equations and Systems

So, what are Diophantine equations? They're just equations with the additional constraint that all solutions must be integers. We've already dealt with the linear Diophantine equation $ax + by = c$, for instance. It is solvable if and only if c is divisible by $\gcd(a, b)$. However, there is no general method to solve most Diophantine equations. In the realm of contest problems, we see that there are a few standard strategies – just be ready to expect the unexpected. A few useful ideas to consider, aside from algebraic manipulations and prime factorizations, include congruences and inequalities. Keep these in mind when tackling the problems.

Problem 1. Find all integers x and y such that $x^2 = 101 + y^2$.

Problem 2. Determine all positive integers m and n such that $mn + 2m + 3n = 2007$.

Problem 3. [AIME 1996 #1] In a magic square, the sum of the three entries in any row, column, or diagonal is the same value. The figure shows four of the entries of a magic square. Find x.

x	19	96
1		

Problem 4. [AIME 1989 #9] One of Euler's conjectures was disproved in the 1980s by three American mathematicians when they showed that there is a positive integer n such that

$$n^5 = 133^5 + 110^5 + 84^5 + 27^5.$$

Find the value of n.

Problem 5. [AIME 1986 #10] In a parlor game, the magician asks one of the participants to think of a three digit number (abc) where a, b, and c represent digits in base 10 in the order indicated. The magician then asks this person to form the numbers (acb), (bca), (bac), (cab), and (cba), to add these five numbers, and to reveal their sum, N. If told the value of N, the magician can identify the original number, (abc). Play the role of the magician and determine (abc) if $N = 3194$.

Counting Solutions

Most of the equations and systems we've dealt with only had a few solutions. Sometimes, when an equation or system has lots of solutions, it is more meaningful to count the number of solutions. Strategies for dealing with these types of problems are a bit different, but the main idea is the same: use the equations to narrow down the possibilities as much as possible. At a certain point, we will know enough to be able to characterize the solutions – either count them, or determine that there are infinitely many. In the latter case, we can usually end up with some parametrization of the families of solutions.

Problem 6. [AMC12A 2004 #3] For how many ordered pairs of positive integers (x, y) is $x + 2y = 100$?

(A) 33 **(B)** 49 **(C)** 50 **(D)** 99 **(E)** 100

Problem 7. [AHSME 1980 #29] How many ordered triples (x, y, z) of integers satisfy the system of equations below?

$$\begin{aligned} x^2 - 3xy + 2yz - z^2 &= 31 \\ -x^2 + 6yz + 2z^2 &= 44 \\ x^2 + xy + 8z^2 &= 100 \end{aligned}$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** a finite number greater than 2 **(E)** infinitely many

Problem 8. [AIME 1993 #4] How many ordered four-tuples of integers (a, b, c, d) with $0 < a < b < c < d < 500$ satisfy $a + d = b + c$ and $bc - ad = 93$?

Not Solving Equations or Systems

A lot of times, Diophantine equations won't have solutions except trivial ones. In that case, there's not much else to do with the problem. Although these problems usually only appear on Olympiad-level contests, the ideas used in solving them are still useful. You may need to solve an auxiliary equation before tackling the problem at hand, for instance. The fact that some equations don't have solutions narrows down the possibilities even more.

Problem 9. Find all solutions in integers to $x^3 + 2y^3 = 4z^3$.

Problem 10. [APMO 1989 #2] Prove that the equation

$$6(6a^2 + 3b^2 + c^2) = 5n^2$$

has no solutions in integers except $a = b = c = n = 0$.

In-Class Problems

Problem 11. [AHSME 1984 #21] Determine the number of triples (a, b, c) of positive integers which satisfy the simultaneous equations

$$\begin{aligned} ab + bc &= 44, \\ ac + bc &= 23. \end{aligned}$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Problem 12. [AIME 1988 #6] It is possible to place positive integers into the vacant twenty-one squares of the 5×5 square shown below so that the numbers in each row and column form arithmetic sequences. Find the number that must occupy the vacant square marked by the asterisk (*).

			*	
	74			
				186
		103		
0				

Problem 13. [AHSME 1985 #26] Four positive integers a, b, c, and d have a product of 8! and satisfy

$$\begin{aligned} ab + a + b &= 524, \\ bc + b + c &= 146, \\ cd + c + d &= 104. \end{aligned}$$

What is $a - d$?

(A) 4 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 12

Problem 14. [AIME 2003I #8] In an increasing sequence of four positive integers, the first three terms form an arithmetic progression, the last three terms form a geometric progression, and the first and fourth terms differ by 30. Find the sum of the four terms.

Problem 15. The integers a, b, c, d satisfy $a > b > c > d$ and

$$a + b + c + d = 2048 \qquad \text{and} \qquad a^2 - b^2 + c^2 - d^2 = 2048.$$

Find the smallest possible value of a.

Problem 16. [AIME 1994 #11] Ninety-four bricks, each measuring $4'' \times 10'' \times 19''$, are to stacked one on top of another to form a tower 94 bricks tall. Each brick can be oriented so it contributes $4''$ or $10''$ or $19''$ to the total height of the tower. How many different tower heights can be achieved using all 94 of the bricks?

Problem 17. [Putnam 1992 A3] For a given positive integer m, find all triples (n, x, y) of positive integers, with n relatively prime to m, which satisfy $(x^2 + y^2)^m = (xy)^n$.

Problem 18. [Poland 1998] Find all integers $a \geq b \geq c \geq 1$ and $x \geq y \geq z \geq 1$ such that

$$a + b + c = xyz \qquad \text{and} \qquad x + y + z = abc.$$

Problems on Vectors and Analytic Geometry

Problem 19. [HMMT 2004] A parallelogram has 3 of its vertices at $(1,2),(3,8)$, and $(4,1)$. Compute the sum of the possible x-coordinates for the 4th vertex.

Problem 20. [HMMT 2003] The graph of $x^4 = x^2y^2$ is the union of n lines. What is the value of n?

Problem 21. [AIME 1990 #7] A triangle has vertices $P = (-8,5), Q = (-15,-19)$, and $R = (1,-7)$. The equation of the bisector of $\angle P$ can be written in the form $ax + 2y + c = 0$. Find $a + c$.

Problem 22. [AIME 1989 #6] Two skaters, Allie and Billie, are at points A and B, respectively, on a flat, frozen lake. The distance between A and B is 100 meters. Allie leaves A and skates at a speed of 8 meters per second on a straight line that makes a 60° angle with AB. At the same time Allie leaves A, Billie leaves B at a speed of 7 meters per second and follows the straight path that produces the earliest possible meeting of the two skaters, given their speeds. How many meters does Allie skate before meeting Billie?

Problem 23. [HMMT 2004] Find the area of the region of the xy-plane defined by the inequality

$$|x| + |y| + |x + y| \leq 1.$$

Problem 24. [AIME 1994 #8] The points $(0,0)$, $(a,11)$, and $(b,37)$ are the vertices of an equilateral triangle. Find the value of ab.

Problem 25. [AIME 1993 #12] The vertices of $\triangle ABC$ are $A = (0,0)$, $B = (0,420)$, and $C = (560,0)$. The six faces of a die are labeled with two A's, two B's, and two C's. Point $P_1 = (k,m)$ is chosen in the interior of $\triangle ABC$, and points P_2, P_3, $P_4, \ldots$ are generated by rolling the die repeatedly and applying the rule: If the die shows label L, where $L \in \{A,B,C\}$, and P_n is the most recently obtained point, then P_{n+1} is the midpoint of $\overline{P_nL}$. Given that $P_7 = (14,92)$, what is $k + m$?

Problem 26. [IMO Shortlist 2001] Let ABC be a triangle with centroid G. Determine, with proof, the position of the point P in the plane of ABC such that $AP{\cdot}AG + BP{\cdot}BG + CP{\cdot}CG$ is a minimum, and express this minimum value in terms of the side lengths of ABC.

Problem Set #1

Problem 27. [AMC12A 2005 #21] How many ordered triples of integers (a, b, c), with $a \geq 2, b \geq 1$, and $c \geq 0$, satisfy both $\log_a b = c^{2005}$ and $a + b + c = 2005$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Problem 28. [AIME 1994 #14] Suppose AB and BC are mirrors of equal length. A beam of light strikes $\overline{BC}$ at point C with angle of incidence $\alpha = 19.94^\circ$ and reflects with an equal angle of reflection. The light beam continues its path, reflecting off line segments $\overline{AB}$ and $\overline{BC}$ according to the rule: angle of incidence equals angle of reflection. Let $\beta = \angle ABC$ and suppose the beam of light originally hits C at an angle α with $\overline{BC}$. Given that $\beta = \alpha/10 = 1.994^\circ$, determine the number of times the light beam will bounce off the two line segments. Include the first reflection at C in your count.

Problem 29. Prove that the product of five consecutive positive integers is never a perfect square.

Problem Set #2

Problem 30. [AIME 1999 #2] Consider the parallelogram with vertices $(10, 45)$, $(10, 114)$, $(28, 153)$, and $(28, 84)$. A line through the origin cuts this figure into two congruent polygons. The slope of the line is m/n, where m and n are relatively prime positive integers. Find $m + n$.

Problem 31. [AIME 2003II #14] Let $A = (0, 0)$ and $B = (b, 2)$ be points on the coordinate plane. Let $ABCDEF$ be a convex equilateral hexagon such that $\angle FAB = 120°$, $\overline{AB} \parallel \overline{DE}$, $\overline{BC} \parallel \overline{EF}$, $\overline{CD} \parallel \overline{FA}$, and the y-coordinates of its vertices are distinct elements of the set $\{0, 2, 4, 6, 8, 10\}$. The area of the hexagon can be written in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

Problem 32. [Korea 1998] Find all pairwise relatively prime positive integers l, m, n such that

$$(l + m + n)\left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right)$$

is an integer.