Systems of Equations

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Solving systems of equations quickly is about exploiting symmetry.

Discussion

The canonical method for solving systems of equations is elimination by substitution. That is, successively solving for a variable and substituting it into the other equations. As an example, consider the system

$$\begin{aligned} x + y + z &= 6 \\ x + 2y + z &= 8 \\ x + y + 3z &= 12. \end{aligned}$$

We would begin the elimination by isolating x in the first equation as $x = 6 - y - z$, then substitute x in the other equations:

$$\begin{aligned} (6 - y - z) + 2y + z &= 6 + y = 8 \\ (6 - y - z) + y + 3z &= 6 + 2z = 12, \end{aligned}$$

discovering the values $y = 2$ and $z = 3$. Backsubstituting then gives $x = 6-y-z = 6-2-3 = 1$, and we have the solution. This strategy is fairly reliable for simple systems of equations, but becomes tedious for large ones. It also fails when we are required to manipulate in more exotic ways; perhaps we cannot even solve for an isolated variable.

This lecture is about combining equations more effectively.

- **Identities.** It is imperative that we are comfortable with some basic factorizations. For any complex numbers[1] x and y, we have

  $$x^2-y^2 = (x-y)(x+y), \quad x^3-y^3 = (x-y)(x^2+xy+y^2), \quad x^3+y^3 = (x+y)(x^2-xy+y^2).$$

[1] That is, numbers of the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$ is the imaginary unit. If you are not familiar with them, you should read about how they multiply in *polar* form, and how they provide two additional (non-real) solutions to $x^3 = 1$.

Example. (HMMT 2007/2) Two reals x and y are such that $x-y=4$ and $x^3-y^3=28$. Compute xy.

Answer: $\boxed{\mathbf{-3}.}$ We have $28 = x^3-y^3 = (x-y)(x^2+xy+y^2) = (x-y)((x-y)^2+3xy) = 4\cdot(16+3xy)$, from which $xy=-3$.

- **Symmetric polynomials.** Define $\mathcal{I}_k$ to be the set of all k-element subsets of $\{1,2,\ldots,n\}$. For variables $x_1,\ldots,x_n$, the *kth elementary symmetric polynomial* is defined by

$$\sigma_k := \sum_{I\in\mathcal{I}_k}\prod_{i\in I} x_i,$$

that is, the sum over all k-term products consisting of terms from $\{x_1,\ldots,x_n\}$. It can be shown that if $P(x_1,\ldots,x_n)$ is *any* symmetric polynomial in $x_1,\ldots,x_n$, then there exists a polynomial $Q(y_1,\ldots,y_n)$ such that

$$P(x_1,\ldots,x_n) = Q(\sigma_1,\ldots,\sigma_n).$$

In particular, if s_k denotes the sum of the kth powers of $x_1,\ldots,x_k$, then

$$\begin{aligned} s_1 &= \sigma_1 \\ s_2 &= \sigma_1^2 - 2\sigma_2 \\ s_3 &= \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3 \\ s_4 &= \sigma_1^4 - 4\sigma_1^2\sigma_2 + 2\sigma_2^2 + 4\sigma_1\sigma_3 - 4\sigma_4, \end{aligned}$$

and so on. The ability to work with symmetric polynomials is so important that we have included it twice.

Example. (HMMT 2007/5) A convex quadrilateral is determined by the points of intersection of the curves $x^4+y^4=100$ and $xy=4$; determine its area.

Answer: $\boxed{\mathbf{4\sqrt{17}}.}$ By symmetry, the quadrilateral is a rectangle having $x=y$ and $x=-y$ as axes of symmetry. Let (a,b) with $a>b>0$ be one of the vertices. Then the desired area is

$$\left(\sqrt{2}(a-b)\right)\cdot\left(\sqrt{2}(a+b)\right) = 2(a^2-b^2) = 2\sqrt{a^4-2a^2b^2+b^4} = 2\sqrt{100-2\cdot 4^2} = 4\sqrt{17}.$$

Note how we did not need to find the exact coordinates of the rectangle. In fact, we could solve for them, but this is a waste of time. In many contests, including the AMC, economy of computation is important in several ways. It is quicker, but it also reduces the drain of a problem and keeps the likelihood of silly errors low.

- **Iteration.** Consider solving for $f(a)$ given that f obeys some equation

$$f(x) + f(g(x)) = h(x),.$$

where g and h are given. It is impossible to find $f(a)$ unless $g^{(k)}(a) = a$ for some positive integer k, where $g^{(k+1)}(x) = g(g^{(k)}(x))$ (one just recovers an unbound system of equations.) Virtually any problem of this form must be solvable by iteration of one sort or another.

Example. (HMMT 2007/8) Let $A := \mathbb{Q} \setminus \{0, 1\}$ denote the set of all rationals other than 0 and 1. A function $f : A \to \mathbb{R}$ has the property that for all $x \in A$,

$$f(x) + f\left(1 - \frac{1}{x}\right) = \log|x|.$$

Compute the value of $f(2007)$.[2]

Answer: $\boxed{\log(\mathbf{2007/2006})}$. Let $g : A \to A$ be defined by $g(x) := 1 - 1/x$; the key property is that

$$g(g(g(x))) = 1 - \frac{1}{1 - \frac{1}{1 - \frac{1}{x}}} = x.$$

The given equation rewrites as $f(x) + f(g(x)) = \log|x|$. Substituting $x = g(y)$ and $x = g(g(z))$ gives the further equations $f(g(y)) + f(g(g(y))) = \log|g(x)|$ and $f(g(g(z))) + f(z) = \log|g(g(x))|$. Setting y and z to x and solving the system of three equations for $f(x)$ gives

$$f(x) = \frac{1}{2} \cdot \left(\log|x| - \log|g(x)| + \log|g(g(x))|\right).$$

For $x = 2007$, we have $g(x) = \frac{2006}{2007}$ and $g(g(x)) = \frac{-1}{2006}$, so that

$$f(2007) = \frac{\log|2007| - \log\left|\frac{2006}{2007}\right| + \log\left|\frac{-1}{2006}\right|}{2} = \log(2007/2006).$$

- **Factorization.** The *fundamental theorem of algebra* holds that any nonconstant polynomial with complex coefficients has a complex root. In particular, we may write

  $$p(x) = x^n + a_1 x^{n-1} + \cdots + a_n = \prod_{i=1}^{n}(x - x_i), \quad (*)$$

  where $x_1, \ldots, x_n$ are the roots of p. Considering the distributive property, we recover *Vieta's formula*

  $$\sigma_k = (-1)^k a_k,$$

  where σ_k is the kth symmetric polynomial in $x_1, \ldots, x_n$. There are many general operations on polynomials: reversing the coefficients, negating every other coefficient, etc. These can analyzed in (*). You might try substituting to eliminate the second highest order term, which can have the effect of forcing a subtle decomposition to become more clear.

[2] If you've never seen log before, you should investigate it at some point. For now, know that $\log x$ defined to be the *real* exponent to which 10 may be raised to equal x (it doesn't exist if $x \leq 0$.) It has various properties, chief among them $\log a + \log b = \log(ab)$.

Problems

1. (AMC10B 2006/12) The lines $x = y/4 + a$ and $y = x/4 + b$ intersect at the point $(1, 2)$. What is $a + b$?

2. (HMMT 2001) Find $x - y$, given that $x^4 = y^4 + 24, x^2 + y^2 = 6$, and $x + y = 3$.

3. (AMC10A 2006/6) What nonzero real value for x satisfies $(7x)^{14} = (14x)^7$?

4. (AMC10A 2004/4) What is the value of x if $|x - 1| = |x - 2|$?

5. (AMC10A 2005/3) The equations $2x + 7 = 3$ and $bx - 10 = -2$ have the same solution x. What is the value of b?

6. (HMMT 2006) Find all real solutions (x, y) of the system $x^2 + y = 12 = y^2 + x$.

7. (AMC10A 2006/8) A parabola with equation $y = x^2 + bx + c$ passes through the points $(2, 3)$ and $(4, 3)$. What is c?

8. (AMC10A 2003/5) Let d and e denote the solutions of $2x^2 + 3x - 5 = 0$. Compute $(d - 1)(e - 1)$.

9. (AMC12A 2007/7) Let a, b, c, d, and e be five consecutive terms in an arithmetic sequence, and suppose that $a + b + c + d + e = 30$. Which of a, b, c, d, or e can be found?

10. (AMC10B 2006/14) Let a and b be the roots of the equation $x^2 - mx + 2 = 0$. Suppose that $a + (1/b)$ and $b + (1/a)$ are the roots of the equation $x^2 - px + q = 0$. What is q?

11. (HMMT 2004) Find all real solutions to $x^4 + (2 - x)^4 = 34$.

12. (AMC12A 2007/17) Suppose that $\sin a + \sin b = \sqrt{\frac{5}{3}}$ and $\cos a + \cos b = 1$. Compute $\cos(a - b)$.

13. (AMC10A 2003/11) The sum of the two 5-digit numbers $AMC10$ and $AMC12$ is 123422. Find $A + M + C$.

14. (HMMT 2002) The real numbers x, y, z, w satisfy

$$\begin{aligned} 2x + y + z + w &= 1 \\ x + 3y + z + w &= 2 \\ x + y + 4z + w &= 3 \\ x + y + z + 5z &= 25. \end{aligned}$$

Find the value of w.

15. (HMMT 2005) Find the sum of the x-coordinates of the distinct points of points of intersection of the plane curves given by $x^2 = x + y + 4$ and $y^2 = y - 15x + 36$.

16. (2006/6) Let a, b, c be the roots of $x^3 - 9x^2 + 11x - 1 = 0$, and let $s = \sqrt{a} + \sqrt{b} + \sqrt{c}$. Find $s^4 - 18s^2 - 8s$.

17. (AIME2 2003/9) The roots of $x^4 - x^3 - x^2 - 1 = 0$ are a, b, c, d. Find $p(a) + p(b) + p(c) + p(d)$, where $p(x) = x^6 - x^5 - x^3 - x^2 - x$.

18. (HMMT 2001) Find the real solutions of $(2x + 1)(3x + 1)(5x + 1)(30x + 1) = 10$.

Homework

1. Let the polynomial $P(x) = x^3 + 14x^2 - 30x + 15$ have roots a, b, and c. Compute the values of $a + b + c$, $a^2 + b^2 + c^2$, and $1/a + 1/b + 1/c$.

   Outline. We read off $\sigma_1 = -14, \sigma_2 = -30, \sigma_3 = -15$. Now

$$\begin{aligned} a + b + c &= \sigma_1 \\ a^2 + b^2 + c^2 &= \sigma_1^2 - 2\sigma_2 \\ 1/a + 1/b + 1/c = (ab + bc + ca)/(abc) &= \sigma_2/\sigma_3. \end{aligned}$$

2. Suppose x, y, and z are complex numbers such that

$$\begin{aligned} x + y + z &= 1 \\ x^2 + y^2 + z^2 &= 3 \\ x^3 + y^3 + z^3 &= 7. \end{aligned}$$

   Find $xy + yz + zx$ and xyz.

   Outline. We are given s_1, s_2, and s_3. We seek the elementary symmetric polynomials σ_2 and σ_3. Use

$$\begin{aligned} s_1 &= \sigma_1 \\ s_2 &= \sigma_1^2 - 2\sigma_2 \\ s_3 &= \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3 \end{aligned}$$

3. (HMMT 2002) Real numbers a, b, c satisfy the equations $a+b+c = 26$, $1/a+1/b+1/c = 28$. Find the value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{a}{c} + \frac{c}{b} + \frac{b}{a}.$$

   Outline. Multiply the two given equations together and subtract three.

4. Let $\mathbb{R}_0$ denote the set of nonzero real numbers. The real valued function f is such that

$$2f(x) + f\left(\frac{1}{x}\right) = x^2 - \frac{1}{x^2}$$

for all $x \in \mathbb{R}_0$. Find $f(2)$.

Outline. Consider plugging in 2 and 1/2. One recovers a system of two equations and two unknowns.

5. Solve for all real x such that

$$\sqrt[3]{38 - x} + \sqrt[3]{38 + x} = 4.$$

Outline. Set $a = \sqrt[3]{38 - x}$ and $b = \sqrt[3]{38 + x}$. Then $a + b = 4$ and

$$76 = a^3 + b^3 = (a + b)(a^2 - ab + b^2) = 4 \cdot (16 - 3ab),$$

so that $ab = -1$. Then solve the quadratic.