Quadratic Equations and the Discriminants

An equation of the type $ax^2 + bx + c = 0$, where a, b, and c are constants and $a \neq 0$, is called the **standard form** of a quadratic equation.

The solutions of any quadratic equation $ax^2 + bx + c = 0$ are given by the **quadratic formula:**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression $b^2 - 4ac$ under the radical sign is called the **discriminant** of the equation.

An equation $ax^2 + bx + c = 0$, with $a \neq 0$ and all coefficients real numbers, has

(A) exactly one real solution if $b^2 - 4ac = 0$;
(B) two distinct real solutions if $b^2 - 4ac > 0$;
(C) no real solutions if $b^2 - 4ac < 0$.

Example 1 One solution of $kx^2 + 3x - k = 0$ is -2. Find the other solution.

2 Prove that the solutions of $ax^2 + bx + c = 0$ are the reciprocals of the solutions of $cx^2 + bx + a = 0$.

3 Solve (a) $x^4 - 9x^2 + 8 = 0$ (b) $x - 3\sqrt{x} - 4 = 0$ (c) $\left(\frac{x^2-2}{x}\right)^2 - 7\left(\frac{x^2-2}{x}\right) - 18 = 0$

(d) $a^3 - 26a^{\frac{3}{2}} - 27 = 0$ (e) $\sqrt{x-3} - \sqrt[4]{x-3} = 2$ (f) $(a^2 - b^2)x^2 - (a^2 + b^2)x - ab = 0$ if $a^2 - b^2 \neq 0$.

(g) $|x^2 - 11x + 10| = |2x^2 + x - 45|$ (h) $\sqrt{x^2 - 5x + 1} + \sqrt{x^2 - 6x + 6} = 2$.

(i) $\sqrt{3x^2 - 5x - 12} - \sqrt{2x^2 - 11x + 15} - x + 3 = 0$ (j) $(x+3)^4 + (x+1)^4 = 82$.

4 Red birds and blue birds, 30 in all, feasted on worms in my yard. The red birds got 108 worms in total, as did the blue birds. Each blue bird got 3 fewer worms than each red bird. How many blue birds were there?

5 What are all real values of x which satisfy $\frac{x^2+2x+1}{x^2+2x+2} + \frac{x^2+2x+2}{x^2+2x+3} = \frac{7}{6}$?

For the quadratic equation $ax^2 + bx + c = 0$, the **sum of the solutions is –b/a**, and the **product of the solutions is c/a**.

6 What are all integers k for which $x^2 + kx + k + 17 = 0$ has integral roots?

7 What are all real values of a for which the sum of the squares of the roots of $x^2 - 8ax + 14a^2 = 0$ is 25?

8 For what integer a are both roots of $x^2 + ax + 17 = 0$ positive integers?

9 What is the least integral value of t for which the roots of the equation $x^2 + 2(t + 1)x + 9t - 5 = 0$ are unequal negative numbers?

10 Let x_1 and x_2 be the solutions of $2x^2 + 5x + 3 = 0$. Find

(a) $x_1^2 + x_2^2$; (b) $\frac{x_2}{x_1} + \frac{x_1}{x_2}$; (c) $(x_1 - 2)(x_2 - 2)$; (d) $(x_1 - x_2)^2$.

11 Given that $19s^2 + 99s + 1 = 0$, $t^2 + 99t + 19 = 0$, and $st \neq 1$. Find the value of $\frac{st + 4s + 1}{t}$.

12 Line with slope 1 through (4, 1), intersects the circle $x^2 + y^2 = 6$ at A and B. Find

(a) the midpoint of the segment $\overline{AB}$; (b) the length of the segment $\overline{AB}$.

Problems:

1 What are all values
(a) of x for which $x(x + 1) = (1991)(1992)$?
(b) of x^5 if $x^{10} + x^5 = 2^{10} + 2^5$?
(c) of x for which $x - 1/x = 5 - 1/5$?

2 Find two numbers whose sum and product are (a) $s = -8$, $p = -33$; (b) $s = \sqrt{2}$, $p = -1/4$.

3 Given $x^2 - 3x - 4 = 0$ (*). Write a equation whose solutions are
(a) the opposites of the solutions of (*); (b) the reciprocals of the solutions of (*);
(c) n times of the solutions of (*); (d) one more than the solutions of (*);
(e) the cubes of the solutions of (*).

4 Solve (a) $(2+\sqrt{3})^{\frac{x}{2}} + (2-\sqrt{3})^{\frac{x}{2}} = 4$ (b) $|x^2 - 2x + 13| = |2x^2 - 3x + 2|$ (c) $x^4 + (x - 4)^4 = 626$

(c) $|x^2 - x| - |5 - 2x| = 1$ (d) $\sqrt{x^2 + 5x - 1} + \sqrt{x^2 - x + 1} = 3x - 1$.

5 Find all values of a such that $\left|x^2 - 5x\right| = a$ has two distinct real solutions.

6 Solve $x + y = x^2 - xy + y^2 + 1$.

7 If p and q are the roots of $2x^2 - 5x + 1 = 0$, what is the value of $\log_2 p + \log_2 q$?

8 What are all ordered pairs of numbers (x, y) which satisfy $x^2 - xy + y^2 = 13$ and $x - xy + y = -5$?

9 What are all real values of a for which the two values of x which satisfy $(a + 1)x^2 - 3ax + 4a = 0$ are unequal numbers, each greater than 1?

10 A student, required to solve the equation $x^2 + bx + c = 0$, inadvertently solves the equation $x^2 + cx + b = 0$; b, c integers. One of the roots obtained is the same as a root of the original equation, but the second root is m less than the second root of the original equation. Find b and c in terms of m.

11 If the solutions of the equation $x^2 + px + q = 0$ are the cubes of the solutions of the equation $x^2 + mx + n = 0$, then

(a) $p = m^3 + 3mn$ (b) $p = m^3 - 3mn$ (c) $p + q = m^3$ (d) $\frac{p}{q} = \left(\frac{m}{n}\right)^3$ (e) none of these

12 If a, b, c, and d are none-zero numbers such that c and d are the solutions of $x^2 + ax + b = 0$ and a and b are the solutions of $x^2 + cx + d = 0$, then a + b + c + d equals

(a) 0 (b) –2 (c) 2 (d) 4 (e) $\frac{-1+\sqrt{5}}{2}$

13 If $\tan\alpha$ and $\tan\beta$ are the two roots of $x^2 - px + q = 0$, $\cot\alpha$ and $\cot\beta$ are the two roots of $x^2 - rx + s = 0$, then rs must be

(a) pq (b) 1/pq (c) p/q^2 (d) q/p^2 (e) p/q

14 Let a and b be two negative real numbers such that $\frac{1}{a} + \frac{1}{b} - \frac{1}{a-b} = 0$. Find the value of $\frac{b}{a}$.

15 If a, b, and c are three distinct real numbers, show that the equations $ax^2 + 2bx + c = 0$, $bx^2 + 2cx + a = 0$, and $cx^2 + 2ax + b = 0$ can't have exactly one real solution simultaneously.

16 Let a, b, and c be three real numbers that satisfy $a + b + c = 0$ and $abc = 1$. Show that one of a, b, and c is greater than 3/2.

17 Show that there are no real numbers x and y such that $1/x + 1/y = 1/(x + y)$.

18 Let a, b, and c be the lengths of the sides of triangle ABC. Show that $b^2x^2 + (b^2 + c^2 - a^2)x + c^2 = 0$ has no real solution.

19 Prove the inequality $(a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2)$, where a_i and b_i are real numbers for i = 1, 2, …, n, and they are equal if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$, $b_i \neq 0$.

20 Let a, b, c, d, and e be real numbers satisfy $a + b + c + d + e = 8$ and $a^2 + b^2 + c^2 + d^2 + e^2 = 16$. Show that all the five numbers are between 0 and 16/5 inclusive.