# Lectures on Challenging Mathematics

# Core Computational Mathematics
# Volume 3.1

# UC3 Algebra





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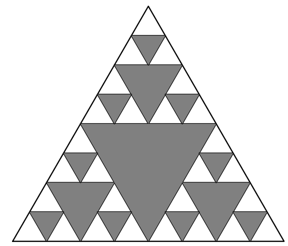
# Contents

## 1.19 Fractals and recursive relations (part 2)

1. An equilateral triangle of unit area is painted step-by-step as follows: Step 1 consists of painting the triangle formed by joining the midpoints of the sides. Step 2 then consists of applying the same midpoint-triangle process to each of the three small unpainted triangles. Step 3 then consists of applying the midpoint-triangle process to each of the nine very small unpainted triangles. The result is shown at right.

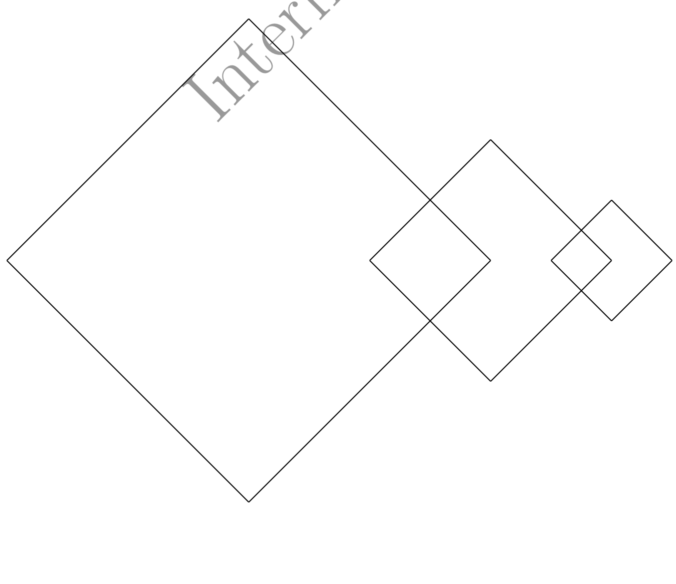


In general, each step consists of applying the midpoint-triangle process to each of the (many) remaining unpainted triangles left by the preceding step. Let $P_n$ be the area that was painted *during* step $n$, and let $U_n$ be the total unpainted area left after $n$ steps have been completed.

   (a) Find $U_1$, $U_2$, $U_3$, $P_1$, $P_2$, and $P_3$.
   (b) Write a recursive description of $U_n$ in terms of $U_{n-1}$. Find an explicit formula for $U_n$.
   (c) Write a recursive description of $P_n$ in terms of $P_{n-1}$. Find an explicit formula for $P_n$.
   (d) Use your work to evaluate the sum $\frac{1}{4} + \frac{3}{16} + \frac{9}{64} + \cdots + \frac{3^{99}}{4^{100}} + \frac{3^{100}}{4^{101}}$.
   (e) Express the series of part (d) using sigma notation.

2. In the sequence $2001, 2002, 2003, \ldots$, each term after the third is found by subtracting the previous term from the sum of the two terms that precede that term. For example, the fourth term is $2001 + 2002 - 2003 = 2000$. What is the $2004^{\text{th}}$ term in this sequence?

3. Square $S_1$ is $1 \times 1$. For $i \geq 1$, the lengths of the sides of square $S_{i+1}$ are half the lengths of the sides of square $S_i$, two adjacent sides of square $S_i$ are perpendicular bisectors of two adjacent sides of square $S_{i+1}$, and the other two sides of square $S_{i+1}$ are the perpendicular bisectors of two adjacent sides of square $S_{i+2}$. Let $\mathcal{R}$ denote region consisting of points lying in at least one of $S_1, S_2, \ldots, S_{10}$. Find the total area of $\mathcal{R}$.



4. Ten guys sit in ten seats in a line. All ten guys get up and then reseat themselves using all ten seats, each sitting in the seat he was in before or a seat next to the one he occupied before. In how many ways can the guys be reseated?

5. Let $H_1 = A_1A_2A_3A_4A_5A_6$ be a regular hexagon with $A_1A_2 = 1$. The common region of equilateral triangles $A_1A_3A_5$ and $A_2A_4A_6$ is another regular hexagon $H_2 = A_1'A_2'A_3'A_4'A_5'A_6'$ where segment $A_1A_3$ intersects segments $A_2A_6$ and $A_2A_4$ at $A_1'$ and $A_2'$, respectively. Triangular regions $A_1A_2A_1'$, $A_2A_3A_2'$, $\ldots$, $A_5A_6A_5'$, $A_6A_1A_6'$ are colored red. Apply a similar procedure to $H_2$, and so on. Find the total area of the colored region.