

Lectures on Challenging Mathematics

Core Computational Mathematics Volume 3.1

UC3 Algebra

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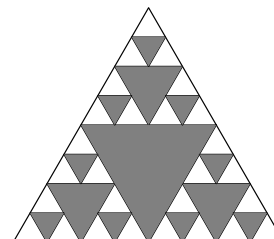
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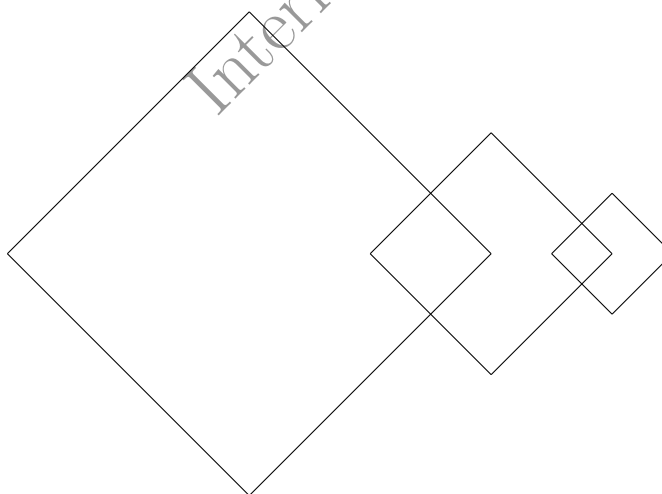
1.19 Fractals and recursive relations (part 2)

1. An equilateral triangle of unit area is painted step-by-step as follows: Step 1 consists of painting the triangle formed by joining the midpoints of the sides. Step 2 then consists of applying the same midpoint-triangle process to each of the three small unpainted triangles. Step 3 then consists of applying the midpoint-triangle process to each of the nine very small unpainted triangles. The result is shown at right.



In general, each step consists of applying the midpoint-triangle process to each of the (many) remaining unpainted triangles left by the preceding step. Let P_n be the area that was painted during step n , and let U_n be the total unpainted area left after n steps have been completed.

- Find $U_1, U_2, U_3, P_1, P_2,$ and P_3 .
 - Write a recursive description of U_n in terms of U_{n-1} . Find an explicit formula for U_n .
 - Write a recursive description of P_n in terms of P_{n-1} . Find an explicit formula for P_n .
 - Use your work to evaluate the sum $\frac{1}{4} + \frac{3}{16} + \frac{9}{64} + \cdots + \frac{3^{99}}{4^{100}} + \frac{3^{100}}{4^{101}}$.
 - Express the series of part (d) using sigma notation.
2. In the sequence 2001, 2002, 2003, \dots , each term after the third is found by subtracting the previous term from the sum of the two terms that precede that term. For example, the fourth term is $2001 + 2002 - 2003 = 2000$. What is the 2004th term in this sequence?
3. Square S_1 is 1×1 . For $i \geq 1$, the lengths of the sides of square S_{i+1} are half the lengths of the sides of square S_i , two adjacent sides of square S_i are perpendicular bisectors of two adjacent sides of square S_{i+1} , and the other two sides of square S_{i+1} are the perpendicular bisectors of two adjacent sides of square S_{i+2} . Let \mathcal{R} denote region consisting of points lying in at least one of S_1, S_2, \dots, S_{10} . Find the total area of \mathcal{R} .



4. Ten guys sit in ten seats in a line. All ten guys get up and then reseal themselves using all ten seats, each sitting in the seat he was in before or a seat next to the one he occupied before. In how many ways can the guys be reseated?
5. Let $H_1 = A_1A_2A_3A_4A_5A_6$ be a regular hexagon with $A_1A_2 = 1$. The common region of equilateral triangles $A_1A_3A_5$ and $A_2A_4A_6$ is another regular hexagon $H_2 = A'_1A'_2A'_3A'_4A'_5A'_6$ where segment A_1A_3 intersects segments A_2A_6 and A_2A_4 at A'_1 and A'_2 , respectively. Triangular regions $A_1A_2A'_1$, $A_2A_3A'_2$, \dots , $A_5A_6A'_5$, $A_6A_1A'_6$ are colored red. Apply a similar procedure to H_2 , and so on. Find the total area of the colored region.