

# Lectures on Challenging Mathematics

## Core Computational Mathematics Volume 2.4

### UC2 Number Sense

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## 1.18 Algebraic techniques in number theory (part 4)

1. The expressions

$$A = 1 \cdot 2 + 3 \cdot 4 + 5 \cdot 6 + \cdots + 37 \cdot 38 + 39$$

and

$$B = 1 + 2 \cdot 3 + 4 \cdot 5 + \cdots + 36 \cdot 37 + 38 \cdot 39$$

are obtained by writing multiplication and addition operators in an alternating pattern between successive integers. Find the positive difference between the integers  $A$  and  $B$ .

2. Let  $k$  be a positive integer. The intersection of the graphs of  $y < k$  and  $y > |x|$  contains at least 90 lattice points. Compute the smallest possible value of  $k$ .
3. Factor each of  $20^2 + 21^2 + \cdots + 40^2$  and  $20^3 + 21^3 + \cdots + 40^3$ .
4. Set  $\{1, 2, 3, 6\}$  satisfies the following conditions:
- it is a four-element set of positive integers that are *relatively prime to each other*; that is, there is no prime that divides each of these four elements;
  - each element divides the sum of all elements in the set;
  - one of the elements is the sum of the other three elements.

Let  $S = \{w, x, y, z\}$  with  $w < x < y < z$  be a set that satisfies conditions (i), (ii), and (iii).

- Show that  $y$  divides  $2w + 2x$ .
  - Show that  $2w + 2x$  is a multiple of  $y$  and the possible values of the quotient  $(2w + 2x)/y$  are 1, 2, 3.
  - Assume that  $2w + 2x = y$ . Find all possible values of  $6w/x$  and find all possible sets  $S$ .
  - Assume that  $2w + 2x = 2y$ . Find all possible values of  $4w/x$  and find all possible sets  $S$ .
  - Assume that  $2w + 2x = 3y$ . Show that  $(10w + x)/3$  is a multiple of  $x$ , determine all possible values of the quotient  $(10w + x)/(3x)$ , and find all possible sets  $S$ .
5. (Continuation) Determine all the sets satisfying conditions (i), (ii), but not (iii).