Lectures on Challenging Mathematics

Core Computational Mathematics
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UC2 Number Sense

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1.18 Algebraic techniques in number theory (part 4)

1. The expressions
$$A = 1 \cdot 2 + 3 \cdot 4 + 5 \cdot 6 + \cdots + 37 \cdot 38 + 39$$
and
$$B = 1 + 2 \cdot 3 + 4 \cdot 5 + \cdots + 36 \cdot 37 + 38 \cdot 39$$
are obtained by writing multiplication and addition operators in an alternating pattern between successive integers. Find the positive difference between the integers A and B.

2. Let k be a positive integer. The intersection of the graphs of $y < k$ and $y > |x|$ contains at least 90 lattice points. Compute the smallest possible value of k.

3. Factor each of $20^2 + 21^2 + \cdots + 40^2$ and $20^3 + 21^3 + \cdots + 40^3$.

4. Set $\{1, 2, 3, 6\}$ satisfies the following conditions:

   (i) it is a four-element set of positive integers that are *relatively prime to each other*; that is, there is no prime that divides each of these four elements;

   (ii) each element divides the sum of all elements in the set;

   (iii) one of the elements is the sum of the other three elements.

   Let $S = \{w, x, y, z\}$ with $w < x < y < z$ be a set that satisfies conditions (i), (ii), and (iii).

   (a) Show that y divides $2w + 2x$.

   (b) Show that $2w + 2x$ is a multiple of y and the possible values of the quotient $(2w + 2x)/y$ are 1, 2, 3.

   (c) Assume that $2w + 2x = y$. Find all possible values of $6w/x$ and find all possible sets S.

   (d) Assume that $2w + 2x = 2y$. Find all possible values of $4w/x$ and find all possible sets S.

   (e) Assume that $2w + 2x = 3y$. Show that $(10w + x)/3$ is a multiple of x, determine all possible values of the quotient $(10w + x)/(3x)$, and find all possible sets S.

5. (Continuation) Determine all the sets satisfying conditions (i), (ii), but not (iii).