Lectures on Challenging Mathematics

RC2
Enhancement of Computational Mathematics Part 2

Summer 2018

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1.3 A touch on the Vandermonde's identity

1. Objects A and B move simultaneously in the coordinate plane via a sequence of steps, each of length one. Object A starts at $(0,0)$ and each of its steps is either right or up, both equally likely. Object B starts at $(5,7)$ and each of its steps is either left or down, both equally likely. What is the probability that the objects meet?

2. Determine the number of 10-digit binary numbers (that is, numbers with only digits 0s and 1s) $\overline{a_1a_2\ldots a_{10}}$ $(a_1 \neq 0)$ such that $a_1+a_3+a_5+a_7+a_9 = a_2+a_4+a_6+a_8+a_{10}$.

3. Seven teams play a soccer tournament in which each team plays every other team exactly once. No ties occur, each team has 50% chance of winning each game it plays, and the outcomes of the games are independent. In each game, the winner is awarded 1 point and the loser gets 0 points. The total points are accumulated to decide the ranks of the teams. In the first game of the tournament, team A beats team B. The probability that team A finishes with more points than team B is m/n, where m and n are relatively prime positive integers. Find $m+n$.

4. (Vandermonde) Let m, n, and k be integers with $m, n \geq 0$. Prove that

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}.$$

5. Determine the number of ways to select a positive number of squares on an 8×8 chessboard such that no two lie in the same row or the same column and no chosen square lies to the left of and below another chosen square.

2.3 Geometry computations and Brahmagupta's formula

1. Quadrilateral $ABCD$ is inscribed in a circle. Given that $AB = a$, $BC = b$, $CD = c$, and $DA = d$. Write $\cos C$ and $\sin C$ in terms of a, b, c, d, respectively.

   Establish the *Brahmagupta's formula*: the area of $ABCD$ is equal to $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ where s denotes the semi-perimeter $(a+b+c+d)/2$.

2. Explain why Heron's formula is a special case of Brahmagupta's formula. *Bretschneiders formula*, which generalizes Brahmaguptas formula, states that the area of an arbitrary quadrilateral with side lengths a, b, c, and d, is given by

   $$\sqrt{(s-a)(s-b)(s-c)(s-d) - abcd(\cos\theta)^2},$$

   where $s = (a+b+c+d)/2$ and θ is half the sum of either pair of opposite angles. Bretschneiders formula can be established in a similar fashion. At this point, we will accept this result without formal proof. Interested readers can explore the proof. What fact can be deduced from the Bretschneiders formula?

3. In rectangle $ABCD$, $AB = 2$ and $AD = x < 2$. Denote by ω the circle centered at A with radius 2. Circle ω intersects ray AD and segment CD at E and F respectively. Let $\mathcal{R}_1$ denote the (convex) region bounded by segments DE, DF and minor arc $\widehat{EF}$, and let $\mathcal{R}_2$ denote the (concave) region bounded by segments CB, CF and minor arc $\widehat{BF}$. Given that $\mathcal{R}_1$ can be cut out and moved to exactly fit in $\mathcal{R}_2$, with D matching C and $\widehat{EF}$ being tangent to $\widehat{BE}$. Find x.

4. Consider all quadrilaterals $ABCD$ such that $AB = 14$, $BC = 9$, $CD = 7$, and $DA = 12$. What is the radius of the largest possible circle that fits inside or on the boundary of such a quadrilateral?

5. Point P lies on side BC of triangle ABC. Suppose that sides AB, BC, CA are have rational lengths. Consider the following statement:

   If both BP and CP have rational lengths, then AP has rational length.

   (a) Determine if the above statement is true.

   (b) What is the converse statement. Is the converse statement true?