Lectures on Challenging Mathematics

RC1
Enhancement of Computational Mathematics Part 1

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6.5 Recursion

1. A collection of ten cubes consists of one cube with edge-length k for each integer k, with $1 \le k \le 10$. A tower is to be built using all ten cubes according to the rules:

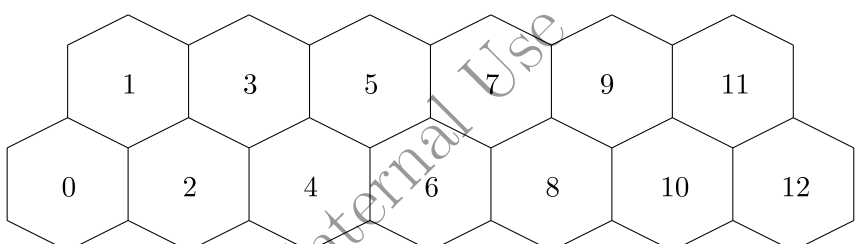
   - Any cube may be the bottom cube of the tower.
   - The cube immediately on top of a cube with edge-length k must have edge-length at most $k+2$.

   Find the number of different towers that can be constructed.

2. Lazy Linus doesn't like doing homework. However, he has to do homework for at least two days for every consecutive three-day block in order not to fail his class. In how many ways can he choose some days (possibly none) out of 14 consecutive days to slack off and not do homework, and still pass his class? Solve this problem by considering the number of days Linus can slack. Then solve it by observing recursion.

3. Consider the honeycomb pattern shown below, which consists of two rows of numbered hexagons. A honeybee crawls from hexagon number 0 to hexagon number 12, always moving from one hexagon to an adjacent hexagon whose number is greater. We want to find the number of different paths for the honeybee to achieve his goal.

   Let $p(n)$ denote the number of paths from hexagon 0 to hexagon n, and assume that $p(0) = 1$. It is easy to see the values of $p(1)$, $p(2)$, and $p(3)$. You should also see a pattern emerging. Establish this pattern.



4. (Continuation) Express $p(n)$ as the sum of binomial coefficients.

5. Let $\{F_n\}_{n=1}^{\infty}$ be the Fibonacci sequence defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \ge 0$. Fibonacci numbers can be obtained as the sums of the binomial coefficients. Construct a Pascal triangle, visualize and prove this result:

$$F_{n+1} = \sum_{k=0}^{n} \binom{n-k}{k}.$$

7.3 Solid geometry (part 2)

1. Jesse cuts a circular paper disk of radius 12 along two radii to form two sectors, the smaller having a central angle of 120 degrees. He makes two circular cones, using each sector to form the lateral surface of a cone. What is the ratio of the volume of the smaller cone to that of the larger?

2. Let points $A = (0, 0, 0)$, $B = (1, 0, 0)$, $C = (0, 2, 0)$, and $D = (0, 0, 3)$. Points E, F, G, and H are midpoints of segments BD, AB, AC, and DC respectively. What is the area of quadrilateral $EFGH$?

3. A solid tetrahedron is sliced off a solid wooden unit cube by a plane passing through two nonadjacent vertices on one face and one vertex on the opposite face not adjacent to either of the first two vertices. The tetrahedron is discarded and the remaining portion of the cube is placed on a table with the cut surface facedown. What is the height of this object?

4. Six solid regular tetrahedra are placed on a surface so that their bases form a regular hexagon $\mathcal{H}$ with side length 1, and so that the vertices not lying in the plane of $\mathcal{H}$ (the *top* vertices) are themselves coplanar. A spherical ball of radius r is placed so that its center is directly above the center of the hexagon. The sphere rests on the tetrahedra so that it is tangent to one edge from each tetrahedron. If the ball's center is coplanar with the top vertices of the tetrahedra, compute r.

5. A unit cube has vertices $P_1, P_2, P_3, P_4, P_1', P_2', P_3'$, and P_4'. Vertices P_2, P_3, and P_4 are adjacent to P_1, and for $1 \le i \le 4$, vertices P_i and P_i' are opposite to each other. A regular octahedron has one vertex in each of the segments $P_1P_2, P_1P_3, P_1P_4, P_1'P_2', P_1'P_3'$, and $P_1'P_4'$. What is the octahedron's side length?