Lectures on Challenging Mathematics

Olympiad Math 1

Number Theory

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1.25 Modular arithmetic (part 5)

1. Consider the the following statement:

   > Let m be a positive integer. Let a be an integer relatively prime to m. If a_1 and a_2 are integers such that $a_1 \not\equiv a_2 \pmod{m}$, then $a_1 a \not\equiv a_2 a \pmod{m}$.

   Determine if the statement is true, find its converse statements, and determine if the converse statement true.

2. Let $P = (1013)(1017)(1113)\cdots(9917)$ be the product of all four-digit numbers that end with either 13 or 17. What are the last three digits of P?

3. The following very famous theorem in number theory belongs to Dirichlet:

   > There are infinitely many primes in any arithmetic sequence of integers for which the common difference is relatively prime to the terms. In other words, let a and m be relatively prime positive integers, then there are infinitely many primes p such that $p \equiv a \pmod{m}$.

   The proof of the general result in this theorem is beyond the scope this work. But, we have learned enough knowledge to prove certain special cases of this theorem.

   Show that there are infinitely many primes of the form

   (a) $4k - 1$ (b) $6k - 1$

   Start your proof with the sentence:

   > Assume, for the sake of contradiction, there are only finitely many primes of the form $4k - 1$ and they are $p_1, p_2, \ldots, p_n$.

4. (Continuation) Can you modify your proof easily to show that there many primes of the form

   (a) $4k + 1$ (b) $6k + 1$

5. When 4444^{4444} is written in decimal notation, the sum of its digits is A. Let B be the sum of the digits of A. Find the sum of the digits of B.