Lectures on Challenging Mathematics

UC3 Practice Tests
An Invitation to Computational Mathematics

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"Cogito ergo Sum" – "I think, therefore I am"

René Descartes (1596-1650)

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Chapter 1

Practice Tests

1.1 Selected entry to medium level algebra problems from AIME

1. Suppose that $|x_i| < 1$ for $i = 1, 2, \ldots, n$. Suppose further that
$$|x_1| + |x_2| + \cdots + |x_n| = 19 + |x_1 + x_2 + \cdots + x_n|.$$
What is the smallest possible value of n?

2. A sequence of integers $a_1, a_2, \ldots$ is chosen so that $a_n = a_{n-1} - a_{n-2}$ for each $n \geq 3$. What is the sum of first 2001 terms of this sequence if the sum of the first 1492 terms is 1985 and the sum of the first 1985 terms is 1492.

3. Let $P_0(x) = x^3 + 313x^2 - 77x - 8$. For integers $n \geq 1$, define $P_n(x) = P_{n-1}(x - n)$. What is the coefficient of x in $P_{20}(x)$?

4. The graphs of $y = 3(x - h)^2 + j$ and $y = 2(x - h)^2 + k$ have y-intercepts of 2013 and 2014, respectively, and each graph has two positive integer x-intercepts. Find h.

5. Real numbers r and s are roots of $p(x) = x^3 + ax + b$, and $r + 4$ and $s - 3$ are roots of $q(x) = x^3 + ax + b + 240$. Find the sum of all possible values of $|b|$.

1.2 Selected entry to medium level number theory problems from AIME (part 1)

1. Find $3x^2y^2$ if x and y are integers such that $y^2 + 3x^2y^2 = 30x^2 + 517$.

2. Suppose n is a positive integer and d is a single digit in base 10. Find n if

$$\frac{n}{810} = 0.d25d25d25\ldots.$$

3. A tennis player computes her *win ratio* by dividing the number of matches she has won by the total number of matches she has played. At the start of a weekend, her win ratio is exactly .500. During the weekend she plays four matches, winning three and losing one. At the end of the weekend, her win ratio is greater than .503. What is the largest number of matches that she cold have won before the weekend began?

4. Let S be the set of all rational numbers r, $0 < r < 1$, that have a repeating decimal expansion of the form

$$0.abcabcabc\cdots = 0.\overline{abc},$$

where digits a, b, c are not necessarily distinct. To write the elements of S as fractions, how many different numerators are required?

5. The numbers in the sequence $101, 104, 109, 116, \ldots$ are of the form $a_n = 100 + n^2$, where $n = 1, 2, 3, \ldots$. For each n, let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.

1.3 Selected entry to medium level number theory problems from ARML (part 1)

1. Let $\overline{abc}$ be a three-digit number. What is the minimum value of $\overline{abc} - (a^2 + b^2 + c^2)$.

2. Suppose \$20.04 is expressed using any combination of pennies, nickels, dimes, and/or quarters with at most 50 of each type of coin. Compute the positive difference between the greatest and least number of coins possible.

3. Let a, b, c, d, and e be distinct elements from the set $\{-10, -9, \ldots, 9, 10\}$. Compute the least value of $a + e$ given the following equations:
$$a - b = 2, \quad c - b = -3, \quad c - d = 4, \quad e - d = -5.$$

4. If, from left to right, the last seven digits of $n!$ are 8000000, compute the value of n.

5. For a positive integer n, let $\langle n \rangle$ denote the closest integral perfect square to n, e.g., $\langle 71 \rangle = 64$ and $\langle 21 \rangle = 25$. Compute the smallest positive integer n such that
$$\langle 91 \rangle \cdot \langle 120 \rangle \cdot \langle 143 \rangle \cdot \langle 180 \rangle \cdot \langle n \rangle = 91 \cdot 120 \cdot 143 \cdot 180 \cdot n.$$

1.4 Selected entry to medium level number theory problems from AIME (part 2)

1. If $a < b < c < d < e$ are consecutive positive integers such that $b + c + d$ is a perfect square and $a + b + c + d + e$ is a perfect cube, what is the smallest possible value of c?

2. Find $x^2 + y^2$ if x and y are positive integers such that

$$xy + x + y = 71 \qquad \text{and} \qquad x^2y + xy^2 = 880.$$

3. A real number r can be expressed as a four-place decimal $0.abcd$, where a, b, c, and d represent digits, any of which could be zero. It is desired to approximate r by a fraction whose numerator is 1 or 2 and whose denominator is an integer. The closest such fraction to r is $\frac{2}{7}$. What is the number of possible values for r?

4. Let k be a given positive integer. Consider the sequence defined by $a_n = k+n^2$ for $n = 1, 2, \ldots$. Find the maximum value of d_n as n ranges through the positive integers.

5. What is the largest 2-digit prime factor of the integer $n = \binom{200}{100}$?

1.5 Selected entry to medium level number theory problems from ARML (part 2)

1. Let a, b, c, and n be positive integers. If $a + b + c = 19 \cdot 97$ and $a + n = b - n = \frac{c}{n}$, find n.

2. Determine the number of three-digit positive integers such that if the integer is divided by the sum of its digits, the result is 19.

3. Let $S = \{1, 2, 3, \ldots, 24, 25\}$. Compute the number of elements in the largest subset of S such that no two elements in the subset differ by the square of an integer.

4. Compute the least prime p such that $p - 1$ equals the difference of the squares of two positive multiples of 4.

5. In the coordinate plane, region $\mathcal{R}$ consists of points (x, y) such that $|\lfloor x \rfloor \cdot \lfloor y \rfloor| = 360$. Find the area of region $\mathcal{R}$.