Lectures on Challenging Mathematics

UC3 Number Theory
An Invitation to Computational Mathematics

Winter 2017

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"Cogito ergo Sum" – "I think, therefore I am"

René Descartes (1596-1650)

Contents

Chapter 1

Number Theory

1.1 Computations with G.C.D. and L.C.M.

1. For a pair of positive integers m and n, recall that their *greatest common divisor*, (denoted by $\gcd(m,n)$) is the greatest integer divides both of them, and that their *least positive common multiple* (denoted by $\text{lcm}(m,n)$) is the least integer multiple of both m and n.

   Compute $\text{lcm}(904, 1288)$. Do you observe any interesting relations among $\gcd(904, 1288)$, $\text{lcm}(904, 1288)$, and $904 \cdot 1288$? Can you prove your observation?

2. If n is a positive integer, then $n!!$ is defined to be $n(n-2)(n-4)\cdots 2$ if n is even and $n(n-2)(n-4)\cdots 1$ if n is odd. For example, $8!! = 8\cdot 6\cdot 4\cdot 2 = 384$ and $9!! = 9\cdot 7\cdot 5\cdot 3\cdot 1 = 945$. Compute the number of positive integers n such that $n!!$ divides $2012!!$.

3. Two farmers agree that pigs are worth \$300 and the goats are worth \$210. When one farmer owes the other money, he pays the debt in pigs or goats, with "change" received in the form of goats or pigs as necessary. (For example, a \$390 debt could be paid with two pigs, with one goat received in change.) What is the amount of the smallest positive debt that can be resolved in this way?

4. In a special football game, a team scores 7 points for a touchdown and 3 points for a field goal. Determine the largest mathematically unreachable number of points scored by a team in an (arbitrarily long) game.

5. If we have $\gcd(m,n) = 1$, and we say that m and n are *relatively prime* (or *coprime*).

   How many positive integers are less than or equal to 200 are relatively prime to

   (a) 15? (b) 24?
   (c) both 15 and 24? (d) either 15 or 24 (or both)?
   (e) either 15 or 24 but not both?

6. Determine the number of

   (a) positive integers k such that $12^{12} = \text{lcm}(6^6, 8^8, k)$.

(b) ordered pairs of positive integers (a, b) such that $\text{lcm}(a, b) = 2^3 5^7 11^{13}$.

7. There is an ample supply of milk in a milk tank. Mr. Fat is given a 5-liter (unmarked) container and a 9-liter (unmarked) container. How can he measure out 2 liters of milk?

8. For each of the following rational expressions, find all positive integers n such that the rational expression cannot be reduced any further.

   (a) $\dfrac{21n+4}{14n+3}$ (b) $\dfrac{8n^2+13n+4}{4n+5}$ (c) $\dfrac{6n^2+5}{3n+1}$

9. How many pairs (x, y) of positive integers are there, with $x \leqslant y$, such that $\gcd(x, y) = 5!$ and $\text{lcm}(x, y) = 50!$?

10. Let $1, 4, \ldots$ and $9, 16, \ldots$ be two arithmetic progressions. The set S is the union of the first 2004 terms of each sequence. How many distinct numbers are in S?

1.2 Divisors

1. Let n be a positive integer such that n is greater than 1200 times any of its prime divisors. Determine the least possible value of n.

2. Compute each of the following.

   (a) $\gcd(15^{15} + 13^{13}, 15^{15} - 3 \cdot 13^{13})$ (b) $\gcd(2^{18} + 1, 2^{24} - 1)$

3. Determine all the positive integers n for which the product of all its positive divisors is equal to 24^{240}.

4. For a positive integer n denote by $\tau(n)$ the number of its divisors. It is clear that
$$\tau(n) = \sum_{d|n} 1.$$
Writing τ in this summation form allows us later to discuss it as an example of a *multiplicative arithmetic function*.
If $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is a prime decomposition of n, prove that
$$\tau(n) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1), \quad \text{and} \quad \prod_{d|n} d = n^{\frac{\tau(n)}{2}} \quad \text{and} \quad \tau(n) \le 2\sqrt{n}.$$

5. If $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is a prime decomposition of n, prove that there are
$$(2a_1 + 1)(2a_2 + 1) \cdots (2a_k + 1)$$
distinct pairs of ordered positive integers (a, b) with $\operatorname{lcm}(a, b) = n$.

6. For a positive integer n denote by $\sigma(n)$ the sum of its positive divisors, including 1 and n itself. It is clear that
$$\sigma(n) = \sum_{d|n} d.$$
Show that if $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is the prime factorization of n, then
$$\sigma(n) = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \cdots \frac{p_k^{\alpha_k+1} - 1}{p_k - 1}.$$

7. Find the sum of

   (a) positive divisors of each of 10000 and 3600;

   (b) even positive divisors of each of 10000 and 9000.

   (c) odd positive divisors of each of 6075 and 472500.

8. Compute the sum of all numbers of the form a/b, where a and b are relatively prime positive divisors of 27000.

9. Find the largest divisor of 1001001001 that does not exceed 10000.

10. For a prime p we say that p^k *fully divides* n and write $p^k \| n$ if k is the greatest positive integer such that $p^k \mid n$. Find n such that 2^n fully divides $3^{1024} - 1$.

1.3 Computational practices (part 1)

1. Zach has chosen five numbers from the set $\{1, 2, 3, 4, 5, 6, 7\}$. If he told Claudia what the product of the chosen numbers was, that would not be enough information for Claudia to figure out whether the sum of the chosen numbers was even or odd. What is the product of the chosen numbers?

2. A positive integer is written on each face of a cube. Each vertex is then assigned the product of the numbers written on the three faces intersecting the vertex. The sum of the numbers assigned to all the vertices is equal to 1001. Find the sum of the numbers written on the faces of the cube.

3. Four positive integers a, b, c, d satisfy

$$\begin{aligned} ab + a + b &= 524, \\ bc + c + b &= 146, \\ cd + c + d &= 104. \end{aligned}$$

   Find all possible values of $a - d$.

4. Compute the number of ordered quadruples of integers (a, b, c, d) satisfying the following system of equations:

$$abc = 12000, \quad bcd = 24000, \quad cda = 36000.$$

5. Find the least positive integer n satisfying the following properties: n is divisible by 3 but not 9, and the sum of n and the product of the digits of n is divisible by 9.

6. Find the smallest positive integer that has exactly 30 positive integer divisors.

7. A pair of positive integers is *golden* if they end in the same two digits. For example $(139, 2739)$ and $(350, 850)$ are golden pairs. What is the sum of all two-digit integers n for which (n^2, n^3) is golden?

8. One of the legs of a right triangle is $2^{34} \cdot 3^{21}$ units long. Both the other leg and the hypotenuse are of integer lengths. How many such triangles are there?

9. Find the last 2-digits in the decimal expansions of each of 19^{90} and 2015^{2015}. (Hint: Binomial expansion could be helpful!)

10. Determine all integers n such that number $f(n)$ is prime for $f(n) = n^4 + 4$.

1.4 Computational practices (part 2)

1. What is the least positive integer that can be expressed as the sum of nine consecutive integers, the sum of ten consecutive integers, and the sum of eleven consecutive integers?

2. How many ordered triples of positive integers (a, b, c) are there for which $a^4b^2c = 54000$?

3. Find all positive integers n that have six divisors (including 1 and itself) and the sum of all divisors of n is equal to 434.

4. Given that x, y are positive integers with x as small as possible, and y minimized with that constraint, and $x(x+1)$ divides $y(y+1)$, but neither x nor $x+1$ divides either y or $y+1$, find $x^2 + y^2$.

5. Compute $\gcd(3^{10} + 3^6 + 2, 3^{15} + 3^{11} + 3^6 + 1)$.

6. Let $N = \overline{\texttt{5AB37C2}}$ be a 7-digit positive integer, where A,B,C are digits between 0 and 9, inclusive. If N is divisible by 792, determine all possible ordered triples (A,B,C).

7. In how many ways can one express 6036 as the sum of at least two (not necessarily positive) consecutive integers?

8. Determine the remainder when $1^3 + 2^3 + \cdots + 2020^3$ is divided by 2016.

9. The number 104060465 is divisible by a five-digit prime number. What is that prime number?

10. For a positive integer n, let $p(n)$ denote the product of the positive integer factors of n. Determine the number of factors n of 2310 for which $p(n)$ is a perfect square.

1.5 Computational practices (part 3)

1. Find the number of positive integers x less than 100 for which
$$3^x + 5^x + 7^x + 11^x + 13^x + 17^x + 19^x$$
is prime.

2. Which 3-digit number has the greatest number of different factors?

3. An n-digit number x has the following property: if the unit digit of x is moved to the front (the leftmost position), the resulting number is equal to $2x$. Find the respective smallest possible values of x and n.

4. Determine all integers n such that number $f(n)$ is prime for $f(n) = n^4 + n^2 + 1$;

5. A subset of the integers $1, 2, \ldots, 100$ has the property that none of its members is 3 times another. What is the largest number of members such a subset can have?

6. Determine the last three digits of each of the decimal expansions of 321^{45} and 789^{10}.

7. Let x and y be positive integers such that $7x^5 = 11y^{13}$. The minimum possible value of x can be written in the form $a^c b^d$, where a, b, c, d are positive integers. Compute $a + b + c + d$.

8. Given that 7999999999 has at most two prime factors, find its largest prime factor.

9. In the following 3×3 array of positive integers, the products of the entries of each row, column, and diagonal are the same. What is the sum of all the possible values of g?
$$\begin{array}{ccc} 50 & b & c \\ d & e & f \\ g & h & 2 \end{array}$$

10. A 4-digit prime p divides 1018065. A 4-digit prime q divides 1003030001. Primes p and q have the same digits. Compute $p + q$.