Lectures on Challenging Mathematics

UC3 Geometry
An Invitation to Computational Mathematics

Winter 2017

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"Cogito ergo Sum" – "I think, therefore I am"

René Descartes (1596-1650)

Contents

Chapter 1

Geometry Knowledge

1.1 Practices in geometry computations (part 1)

1.1.1 Mixed exercises 1

1. Two tangents to a circle are drawn from a point A. The points of contact B and C divide the circle into arcs with lengths in the ratio $2 : 3$. What is the degree measure of $\angle BAC$?

2. A unit square is rotated 30° counterclockwise about one of its vertices. Determine the area of the intersection of the original square with the rotated one.

3. Three one-inch squares are placed with their bases on a line forming a 3×1 rectangle. The center square is lifted out and rotated 45°. Then it is centered and lowered into its original location until it touches both the adjoining squares. Point B lies on the boundary on the center square and is the furthest away from the line on which the bases of the original squares were placed. What is the distance, in inches, from B to the base line?

4. Rhombus $ABCD$ has side length 2 and $\angle B = 120^\circ$. Region $\mathcal{R}$ consists of all points inside the rhombus that are closer to vertex B than any of the other three vertices. What is the area of $\mathcal{R}$?

5. Point P lies inside equilateral triangle ABC. Points Q, R, S are the feet of the perpendiculars from P to sides AB, BC, CA, respectively. Given that $PQ = 1$, $PR = 2$, and $PS = 3$, compute the area of the triangle.

1.1.2 Revisiting special angles (part 1)

6. A triangle has a 60-degree angle and a 45-degree angle, and the side opposite the 45-degree angle is 240 units long. How long is the side opposite the 60-degree angle?

7. In rectangle $ABCD$, $DC = 2CB$ and points E and F lie on side AB so that $\angle ADE = \angle EDF = \angle FDC = 30^\circ$. What is the ratio of the area of triangle DEF to the area of rectangle $ABCD$?

8. A square in the coordinate plane has vertices whose y-coordinates are 0, 1, 4, and 5. What is the area of the square?

9. Let ω be a circle with center O, and A and B be two points on circle ω with $\angle AOB = 60^\circ$. Let M be a point on minor arc $\widehat{AB}$. Line ℓ_1 passes through the midpoints of segments AM and OB. Line ℓ_2 passes through the midpoints of segments BM and OA. Find the angle formed by lines ℓ_1 and ℓ_2.

10. In triangle ABC, $AB = 14$, $BC = 16$, and $CA = 26$. Let M be the midpoint of side BC, and let D be a point on segment BC such that AD bisects $\angle BAC$. Compute PM, where P is the foot of perpendicular from B to line AD.

1.2 Practices in geometry computations (part 2)

1.2.1 Mixed exercises 2

1. Let $ABCD$ be a rectangle with $AB = 6$ and $BC = 4$. Let E be the point on side BC with $BE = 3$, and let F be the point on segment AE such that F lies halfway between the segments AB and CD. If G is the point of intersection of lines DF and BC, find BG.

2. Two concentric circles have radii 1 and 2. Two points on the outer circle are chosen independently and uniformly at random. What is the probability that the chord joining the two points intersects the inner circle?

3. Charlie folds an $\frac{17}{2} \times 11$-inch piece of paper in half twice, each time along a straight line parallel to one of the papers edges. What is the smallest possible perimeter of the piece after two such folds?

4. Let ABC be an equilateral triangle with $AB = 3$. Circle ω with diameter 1 is drawn inside the triangle such that it is tangent to sides AB and AC. Let P be a point on ω and Q be a point on segment BC. Find the minimum possible length of the segment PQ.

5. Equilateral triangle XYZ is inscribed in a unit circle ω. Let W be a point other than X in the plane such that triangle WYZ is also equilateral. Determine the area of the region inside triangle WYZ that lies outside circle ω.

1.2.2 Revisiting the centers of a triangle

6. (a) Square $ABCD$ has 8-inch sides, M is the midpoint of BC, and N is the intersection of AM and diagonal BD. Find the length of AN by
   (1) viewing BN as the bisector of $\angle ABM$;
   (2) viewing N as the centroid of a triangle;
   (3) spotting a pair of similar triangles.

   (b) Parallelogram $PQRS$ has $PQ = 8$ cm, $QR = 9$ cm, and diagonal $QS = 10$ cm. Mark F on RS, exactly 5 cm from S. Let T be the intersection of PF and QS. One of the method in part (a) can be adapted to find the lengths TS. Which one? Why the other two methods do not work any more? Find the lengths of TS and TQ.

7. In triangle ABC, $AB = AC = 17$ and $BC = 16$.

   (a) Find the distance from A to the centroid of triangle ABC;
   (b) Find the distance from A to the circumcenter of triangle ABC.
   (c) Find the distance from A to the orthocenter of triangle ABC. (Hmm, what interesting fact can you observe from these three distances?)
   (d) Find the distance from A to the incenter of triangle ABC.

(f) Triangle ABC lies inside a circle (that is, it lies within the boundary of the circle), how small can the radius of the circle be? A circle lies inside triangle ABC (that is, the circle lies within the boundary of the triangle), how large can the radius of the circle be?

8. (Continuation) What if $AB = AC = 15$ and $BC = 24$?

9. In triangle ABC, $AB = 12, BC = 5$, and $CA = 13$. Find the distance between the centroid and orthocenter of the triangle.

10. (Continuation) Find the distance between the incenter and the circumcenter of the triangle.

1.3 Practice in geometry calculations (part 3)

1.3.1 Mixed exercises 3

1. Circles $\omega_1, \omega_2, \omega_3$, each have radius 1, and are centered at A, B, and C, respectively. Circles ω_1 and ω_2 share one point of tangency. Circle ω_3 has a point of tangency with the midpoint of segment AB. What is the area inside circle ω_3 but outside circle ω_1 and circle ω_2?

2. Riders on a Ferris wheel travel in a circle in a vertical plane. A particular wheel has a radius 20 feet and revolves at the constant rate of one revolution per minute. How many seconds does it take a rider to travel from the bottom of the wheel to a point 10 vertical feet above the bottom?

3. Triangle ABC has $AB = 3$, $BC = 4$, and $AC = 5$. The points D, E, and F are the midpoints of sides AB, BC, and AC, respectively. Let $X \neq E$ be the intersection of the circumcircles of triangles BDE and CEF. What is $XA + XB + XC$?

4. In triangle ABC, $\angle C = 90^\circ$. Point P lies on segment BC and is not B or C. Point I lies on segment AP. If $\angle BIP = \angle PBI = \angle CAB = m^\circ$ for some positive integer m, find the sum of all possible values of m.

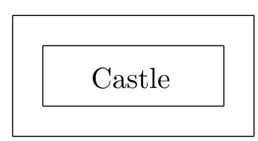
5. Two points on the circumference of a circle of radius r are selected independently and at random. From each point, a chord of length r is drawn in a clockwise direction. What is the probability that the two chords intersect?

1.3.2 Revisiting special angles (part 2)

6. Three congruent isosceles triangles are constructed with their bases on the sides of an equilateral triangle of side length 1. The sum of the areas of the three isosceles triangles is the same as the area of the equilateral triangle. What is the length of one of the two congruent sides of one of the isosceles triangles?

7. Mark P inside square $ABCD$, so that triangle ABP is equilateral. Let Q be the intersection of BP with diagonal AC. Triangle CPQ looks isosceles. Is this actually true? If $AB = 1$, find the area of triangle APC.

8. As shown in the diagram on the right, a castle is surrounded by a rectangular moat, which is of uniform width 12 feet. The problem is to get across the moat to the castle from the dry land on the other side, without being able to use the drawbridge. All you have to work with are two rectangular planks, whose lengths are 11 feet and 11 feet, 9 inches. Find a way to get across.



9. In rectangle $ABCD$, $AB = 20$ and $BC = 10$. Let E be a point on CD such that $\angle CBE = 15^\circ$. What is AE?

10. Let ABC be an equilateral triangle. Semicircle ω, with segment BC as its diameter, is constructed outside of the triangle. Points K and L trisects semicircle ω. Points P and Q are the intersections of segments BC with AK and AL, respectively. Compute PQ/BC.

1.4 Practices in geometry computations (part 4)

1.4.1 Mixed exercises 4

1. Two circles of radius 2 are centered at $(2, 0)$ and $(0, 2)$. What is the area of the intersection of the interiors of the two circles?

2. Let $ABCDEF$ and $ABMNOP$ be two connected regular hexagons of side length 1. Compute the area of quadrilateral $PNCE$.

3. In triangle ABC, $AB = BC = 1$ and $AC = \sqrt{2}$. Segment AB is a diameter of circle ω. Semicircle γ has its center lying on side BC. Given that γ is tangent to ω and γ passes through C, find the radius of γ.

4. In rectangle $ABCD$, $AB = 6$, $AD = 30$, and G is the midpoint of side AD. Segment AB is extended 2 units beyond B to point E, and F is the intersection of segments ED and BC. What is the area of $BFDG$?

5. What is the radius of the largest circle that you can draw on graph paper that encloses $n = 1$ lattice points? How about $n = 2, 3, 4$?

1.4.2 Area, similarity, and Ceva (part 1)

6. The sides of triangle ABC are $AB = 13$, $BC = 15$, and $CA = 14$. Point P is located within the triangle so that triangle PAB has area 28, triangle PBC has area 35, and triangle PCA has area 21. If AP is extended until it meets BC at K, what is the length of BK? If CP is extended until it meets AB at L, what is the length of AL?

7. Let $A = (0, 0)$, $B = (6, 0)$, and $C = (0, 6)$. Point P lies inside the triangle.
   (a) Describe all points P such that the area of triangle ABP is equal to 9.
   (b) Describe all points P such that the area of triangle CAP is equal to 3.
   (c) Describe all points P such that the area of triangle BCP is equal to 6.
   (d) Find point P such that the area ratio between triangles ABP, BCP, CAP is $[ABP] : [BCP] : [CAP] = 3 : 2 : 1$. Find the respective ratios into which the line AP divides the side BC, the line BP divides the side CA, and the line CP divides the side AB.

8. Let P be a point inside triangle ABC. Line AP intersects side BC in D, line BP intersects side CA in E, and line CP intersects side AB in F. Suppose that $DP = 2$, $PA = 5$, $AF = 4$, $FB = 3$.
   (a) Construct point Q on line CF such that $DQ \parallel AB$. Find two pairs of similar triangles involving point Q and compute BD/CD.
   (b) Construct point R on line CF such that $BR \parallel AD$. Find two pairs of similar triangles involving point R and compute BD/CD.

9. Equilateral triangles ABE, BCF, CDG, and DAH are constructed outside the unit square $ABCD$. Eliza wants to stand inside octagon $AEBFCGDH$ so that she can see every point in the octagon without being blocked by a side of the octagon. What is the area of the region in which she can stand?

10. In triangle ABC, $\angle C = 90^\circ$, $AC = 10$, and $BC = 12$. Semicircle ω, with AC as its diameter, is erected outside of the triangle. Let P denote the midpoint of ω. Segment BP cuts the whole region (bounded by sides BA, BC, and ω) into two pieces. What is the positive difference between the areas of the two pieces?

1.5 Power-of-a-point theorem (part 1)

1.5.1 Power-of-a-point theorem (part 1)

1. Suppose that chords AB and CD meet, when extended (through B and D respectively), at point P outside their circle. Given measurements $AB = 10$, $BP = 8$, and $DP = 9$, calculate CD.

2. The following problem was given on a recent national level contest:

   > Quadrilateral $KLMN$ is inscribed in circle ω. The midpoint O of diagonal NL is also the center of ω. Diagonal KM meets segment OL in P. Given that $PL = 8$, $OP = 2$, $KN = 18$, and $PK = 9$, find the length of segment MN.

   Find the length of segment MN by first identifying a triangle that is similar to triangle PNK, and then finding the length of segment LM.

3. (Continuation) Find MN by first finding KL by working with triangle KNL, and then identifying a triangle that is similar to triangle PLK.

   Hmm... By now, you should know that the problem is incorrect. But why it is incorrect?

4. Quadrilateral $KLMN$ is inscribed in circle ω. The midpoint O of diagonal NL is also the center of ω. Diagonals KM and LN meet in P. Given that $PL = 8$, $PN = 12$, and $LM = 16$, find KP.

5. A circle intersects the x-axis at $A = (2, 0)$ and $B = (3, 0)$ and intersects the y-axis at $C = (0, 4)$ and $D = (0, a)$. Determine the value of a.

1.5.2 Revisiting special angles (part 3)

6. A circle is circumscribed about a triangle with sides 20, 21, and 29, thus dividing the interior of the circle into four regions. Let A, B, and C be the areas of the non-triangular regions, with C being the largest. Then which of the following relations is correct:

   (1) $A + B = C$ (2) $A + B + 210 = C$ (3) $A^2 + B^2 = C^2$

   (4) $20A + 21B = 29C$ (5) $\frac{1}{A^2} + \frac{1}{B^2} = \frac{1}{C^2}$

7. A square and an equilateral triangle together have the property that the area of each is the perimeter of the other. Find the square's area.

8. Carefully draw an equilateral triangle with side length 5 and a triangle with sides $3, 5$, and 7.

   (a) It seems that these two triangle can be put together to form another triangle. Try to do so.

   (b) Show that there is a familiar angle in a triangle with sides $3, 5$, and 7.

(c) Show that there is a familiar angle in a triangle with sides $5, 7$, and 8.

9. A square of side length 1 and a circle of radius $\sqrt{3}/3$ share the same center. What is the area inside the circle, but outside the square?

10. A circle with center O has area 156π. Triangle ABC is equilateral, segment BC is a chord on the circle, $OA = 4\sqrt{3}$, and point O lies outside triangle ABC. What is the side length of triangle ABC?

1.6 Practice in geometry calculations (part 5)

1.6.1 Mixed exercises 5

1. What is the area of the region bounded by the graphs of $y = |x+2| - |x-2|$ and $y = |x+1| - |x-3|$.

2. The y-intercepts, P and Q, of two perpendicular lines intersecting at the point $A = (6, 8)$ have a sum of zero. What is the area of triangle APQ?

3. Let $ABCDE$ be a convex pentagon with $DC \parallel AB$ and $AE \perp ED$. Given that $\angle BAE = \angle BCD$ and $\angle D = 130^\circ$, compute $\angle B$.

4. Line ℓ bisects the line $3y = 4x$ and the line $5y = 12x$. Find the slope of ℓ by applying
   (a) the Angle Bisector Theorem in the triangle with vertices $(0, 0), (15, 20)$, and $(15, 36)$.
   (b) the following fact from geometry: In an isosceles triangle, the median to the base bisects the vertex angle. (By the way, how would you restate this fact in a rhombus?)

5. Let $ABCD$ be a square with $AB = 5$. Points E, F, G, H lie on sides AB, BC, CD, DA, respectively with $AE = BF = CG = DH = 2$. What is the area of the square inclosed by segments AF, BG, CH, DE?

1.6.2 Area, similarity, and Ceva (part 2)

6. A triangle with vertices $(6, 5)$, $(8, -3)$, and $(9, 1)$ is reflected about the line $x = 8$ to create a second triangle. What is the area of the union of the two triangles?

7. Let $A = (0, 0)$ and $B = (0, 8)$. Plot several points P that make angle APB a 30-degree angle. Use a protractor, and be prepared to report coordinates for your points. (In case you are familiar with it, your configuration should look like the MasterCard logo). Explain. By the way, AB is called the *common chord* of the two circles. What is the area enclosed by this configuration?

8. A close look at a color television screen reveals an array of thousands of tiny red, green, and blue dots. This is because any color can be obtained as a *mixture* of these three colors. For example, if neighboring red, green, and blue dots are equally bright, the effect is white. If a blue dot is unilluminated and its red and green neighbors are equally bright, the effect is yellow. In other words, white corresponds to the red : green : blue ratio $\frac{1}{3} : \frac{1}{3} : \frac{1}{3}$ and pure yellow corresponds to $\frac{1}{2} : \frac{1}{2} : 0$. Notice that the sum of the three terms in each proportion is 1. A triangle RGB provides a simple model for this mixing of colors. The vertices represent three neighboring dots. Each point C inside the triangle represents a precise color, defined as follows: The *intensities* of the red dot, green dot, and blue dot are proportional to the *areas* of the triangles CGB, CBR, and CRG, respectively.

(a) What color is represented by the centroid of RGB?

(b) What color is represented by the midpoint of side RG?

(c) Point C is $\frac{3}{5}$ of the way from R to G. Give a numerical description for the color mixture that corresponds to it.

(d) The color *magenta* is composed of equal intensities of red and blue, with green absent. Where is this color in the triangle?

(e) Given that color $\mathcal{C}$ is defined by the red : green : blue ratio $0.4 : g : b$, where $g + b = 0.6$, what are the possible positions for $\mathcal{C}$ in the triangle?

9. In triangle RGB, mark P on side RB so that $RP : PB = 3 : 2$. Let C be the midpoint of GP. Calculate $[CGB] : [CBR] : [CRG]$ the ratio of areas of triangles CGB, CBR, CRG. Express your answer in each of the following two different ways:

(a) so that the sum of the three numbers is 1;

(b) so that the three numbers are all whole numbers.

10. In triangle RGB, point X divides RG according to $RX : XG = 3 : 5$, and point Y divides GB according to $GY : YB = 2 : 7$. Let C be the intersection of BX and RY.

(a) Find a ratio of whole numbers that is equal to the area ratio $[CGB] : [CBR]$.

(b) Find a ratio of whole numbers that is equal to the area ratio $[CBR] : [CRG]$.

(c) Find a ratio of whole numbers that is equal to the area ratio $[CGB] : [CRG]$.

(d) Find whole numbers m, n, and p so that $[CGB] : [CBR] : [CRG] = m : n : p$.

(e) The line GC cuts the side BR into two segments. What is the ratio of their lengths?

1.7 Practice in geometry calculations (part 6)

1.7.1 Mixed exercises 6

1. A dart board is a regular octagon $A_1A_2A_3A_4A_5A_6A_7A_8$ divided into regions by diagonals A_1A_6, A_2A_5, A_3A_8, A_4A_7. Suppose that a dart thrown at the board is equally likely to land anywhere on the board. What is the probability that the dart lands within the center square?

2. Let ABC be a triangle, and let D, E, F be the midpoints of sides BC, CA, AB, respectively. The angle bisectors of $\angle FDE$ and $\angle FBD$ meet at P. Given that $\angle BAC = 37^\circ$ and $\angle CBA = 85^\circ$, determine the degree measure of $\angle BPD$.

3. Consider a (not necessarily regular) convex heptagon $\mathcal{P} = A_1A_2A_3A_4A_5A_6A_7$. There are two natural ways to construct a 7-point star from these seven vertices of the heptagon. One of them is $\mathcal{P}_1 = A_1A_3A_5A_7A_2A_4A_6$. Determine with justification the sum of vertex angles of $\mathcal{P}_1$.

   What if we work with the other 7-point star $\mathcal{P}_2 = A_1A_4A_7A_3A_6A_2A_5$?

4. Four congruent circular arcs (denoted by $\widehat{AB}, \widehat{BC}, \widehat{CD}, \widehat{DA}$) of measure 270° are connected at their endpoints (A, B, C, D). The radius of the arcs is equal to 2, what is the area of the region enclosed by the four arcs?

5. Suppose points A and B lie on a circle of radius 4 with center O, such that $\angle AOB = 90^\circ$. The perpendicular bisectors of segments OA and OB divide the interior of the circle into four regions. Find the area of the smallest region.

1.7.2 Revisiting arcs and angles

6. Regular heptagon $HEXAGON$ is inscribed in a circle. What are the angles in the triangle AGE?

   Let $P_1, P_2, \ldots, P_n$ are evenly distributed around a circle. Determine the minimum value of n, given that there are three points P_i, P_j, P_k such that in triangle $P_iP_jP_k$

   (a) $\angle P_i = \dfrac{180^\circ}{7}$, $\angle P_j = \dfrac{360^\circ}{7}$, $\angle P_k = \dfrac{720^\circ}{7}$ (b) $\angle P_i = 40^\circ$, $\angle P_j = 60^\circ$, $\angle P_k = 80^\circ$

7. Polygon $A_1A_2\ldots A_n$ is a regular n-gon. For some integer $k < n$, quadrilateral $A_1A_2A_kA_{k+1}$ is a rectangle of area 6. If the area of $A_1A_2\ldots A_n$ is 60, compute n.

8. In triangle ABC, we have $AB = 7$, $AC = 8$, and $BC = 9$. Point D lies on the circumscribed circle of the triangle so that ray AD bisects $\angle BAC$. What is the value of AD/CD?

9. Let M and A be two given points on circle ω with minor arc $\widehat{MA} = 80^\circ$. Let T and H be two moving points on the major $\widehat{MA}$ with minor arc $\widehat{TH} = 100^\circ$. Chords AH and MT meet at P. As T and H moving along the arc, what is the locus of P?

10. Distinct points A and B are on a semicircle with diameter MN and center C. Point P lies on segment CN and $\angle CAP = \angle CBP = \alpha$ and $\angle ACM = \beta$. Express $\angle BPN$ in terms of α and β.

1.8 Practice in geometry calculations (part 7)

1.8.1 Mixed exercises 7

1. Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle ADC = 90°$. Explain why $ABCD$ is cyclic and determine O the center of its circumcircle. Let diagonals AC and BD intersect at P. Assume that $AC = 20$, $BP = 7$, $DP = 9$, find OP.

2. Points C and D lie on the same side of line AB such that $AB = 25$, $AC = 15$, $AD = 24$, $BC = 20$, and $BD = 7$. Given that rays AC and BD intersect at point E, compute $EA+EB$.

3. Compute the area of the region defined by $x^2 + y^2 \leq 4|x| + 4|y|$.

4. Let ABC be a right triangle, $\angle C = 90°$. The angle-bisector of angle C intersects the perpendicular bisector of AB in S. Show that $ABCS$ is a cyclic quadrilateral.

5. In rectangle $ABCD$, points E and F are on sides AB and CD, respectively, such that $AE = CF > AD$ and $\angle CED = 90°$. Lines AF, BF, CE and DE enclose a rectangle whose area is 24% of the area of $ABCD$. Compute BF/CE.

1.8.2 Area, similarity, and Ceva (part 3)

6. The arcs of four quarter-circles are drawn inside a circle intersecting the circle in pairs of points: (A, B), (B, C), (C, D), (D, A), respectively. If all of the five circles have radius 1 cm, what is the area, in square centimeters, of the region enclosed by the four quarter-circles?

7. Construct, with only a unmarked straight edge, a triangle RGB (of your choice, certainly *no need to be equilateral*) with an interior point P such that such that the area ratio of triangles PGB, PBR, PRG is $[PGB] : [PBR] : [PRG] = 2 : 3 : 5$.

   Accurate drawing is the theme of the problem. You should use the *graph paper* well and choose your triangle (not necessarily a equilateral triangle at all) wisely to avoid any unnecessary estimations. You can only draw lines with the straight edge, and you cannot not estimate any non-lattice points unless they are the intersection of the grid lines or the lines constructed by the straight edge.

8. In triangle RGB, point X divides side RG according to $RX : XG = m : n$, and point Y divides side GB according to $GY : YB = p : q$. Let C be the intersection of segments BX and RY. Find the area ratios

   (a) $[CGB] : [CBR]$; (b) $[CBR] : [CRG]$;
   (c) $[CGB] : [CRG]$; (d) $[CGB] : [CBR] : [CRG]$.

   Also find the ratio into which the line GC divides the side BR.

9. Mixtures of *three* quantities can be modeled geometrically by using a triangle.

   (a) What geometric figure would be suitable for describing the mixing of *two* quantities?

(b) Kirby wants to use a point inside a unit square to describe the mixing of *four* quantities. After some careful thought, he declares that this method does not work and lists a few reasons. Which of these reasons are really legit?

- It is hard to assign a triangle to a quantity. In particular, there is no triangle that is *opposite* to a quantity.
- No triangle can represent a quantity that is more than 0.5.
- Two of the triangles together always represent a total quantity of 0.5.

(c) What geometric figure would be suitable for describing the mixing of *four* quantities? Give the details of your models.

10. In rectangle $ABCD$, $AB = 11$ and $BC = 13$. Points P and Q lie on sides AB and AD respectively with $BP = 2$. Diagonal BD meets segments CP and CQ in X and Y, respectively. Given that the area of triangle CXY is the equal to the sum of the areas of triangles BPX and DQY, find the length of segment DQ.

1.9 Power-of-a-point theorem (part 2)

1.9.1 Power-of-a-point theorem (part 2)

1. In quadrilateral $ABED$, rays AD and BE meet at C. Suppose that $\angle ABE = \angle CDE$. Given that $DA = 2$, $CE = 4$, $EB = 8$, and $CD = x$. Find x by showing that $x(x + 2) = 4 \cdot 12$.

   Note that $x = -8$ is a also an solution to $x(x+2) = 4 \cdot 12$. We are trying to find a configuration to explain this fact. Consider a self-intersecting quadrilateral $ABED$ with sides AB and DE intersecting each other. We also consider the concept of *directed length*. In particular, assigne one direction to CA and CD, and the opposite direction to DA; that is, $DA = -AD$. Explain the rest of the details.

2. Let ω be a circle with center O and radius R, and let X be a point in the plane that is outside of ω. A line is drawn through X; it intersects ω at Y and Z. (If the line is tangent to the circle, then $Y = Z$). Prove that $XY \cdot XZ = XO^2 - R^2$ by either

   (a) constructing the line XT that is tangent to ω at T; or

   (b) constructing the line XO.

3. (Continuation) A similar result can be established when X is inside the circle. State and prove this result. Sometimes, this is called the *cross-chord theorem*.

4. Putting the results in the previous two problems together together, we have the following statement:

   > Let ω be a circle with center O and radius R, and let X be a point in the plane. A line is drawn through X; it intersects ω at Y and Z. (If the line is tangent to the circle, then $Y = Z$). Then the product $XY \cdot XZ = XO^2 - R^2$ is constant (it does not depend on the line drawn), and it is called the *power* of X with respect to ω. (The lengths are directed.)

   (a) Given the circle ω in the plane, determine the respective regions of all points in the plane that has positive, negative, zero power respect to ω.

   (b) Restate a simplified version of the statement by complete the following sentence.

   > Let ω be a circle, and let X be a point in the plane. Two lines are drawn through X. If one line intersects ω at Y and Z and the other line intersects ω at P and Q, then ____________________.

   (c) State the converse of the statement. Is the converse statement true? One needs to be a bit careful about configurations. (Hint:
   This inverse together with the original statement is called the *Power-of-a-point theorem*.

5. Find all points Q in the coordinate plane such that there are exactly two circles passing through points $P = (2, 3)$ and Q, both tangent to the x-axis.

1.9.2 Tangent circles

6. We are given two lines that are perpendicular to each other. A circular disc of radius 1 is placed so that it touches both of these lines. A larger circular disc of radius r is then placed so that it touches the smaller disc and both of these lines. Compute r.

7. Let XYZ be a triangle with $\angle XYZ = 40^\circ$ and $\angle YZX = 60^\circ$. A circle ω, centered at the point I, lies inside triangle XYZ and is tangent to all three sides of the triangle. Let A be the point of tangency of ω with side YZ, and let ray XI intersect side YZ at B. Determine the measure of $\angle AIB$.

8. A square region $ABCD$ is externally tangent to the circle with equation $x^2 + y^2 = 1$ at the point $(0, 1)$ on the side CD. Vertices A and B are on the circle with equation $x^2 + y^2 = 4$. What is the side length of this square?

9. Circles with centers A and B have radii 1 and 2, respectively. The distance between centers of circles is 6. Find the radius of a circle that is tangent to both of these circles and to the segment AB.

10. Circles of radii $5, 5$ and 8 are mutually externally tangent to each other. A smaller circle with radius m/n, where m and n are relatively prime positive integers, is placed so that it is externally tangent to all of these circles. Find $m + n$.

1.10 Practice in geometry calculations (part 8)

1.10.1 Mixed exercises 8

1. Tracey is about to attempt a 5-foot putt on a level surface. The hole is 4 inches in diameter. Remembering the advice of a golf pro, Tracey aims for a mark that is 6 inches from the ball and on the line from the center of the hole to the center of the ball. Tracey misses the mark by a fifth of an inch. Eric claims that, in the ideal situation, Tracey will miss the putt. What do you think about Eric's claim?

2. Point D lies side of AB of triangle ABC. Point F lies on segment CD such that $\angle ABF = \angle CBF$. Give that $AD = 3$, $DB = 5$, and $BC = 10$, find the ratio into which the line AF divides the side BC.

3. Let PQR be a triangle with $\angle P = 75^\circ$ and $\angle Q = 60^\circ$. A regular hexagon $ABCDEF$ with side length 1 is drawn inside triangle PQR so that side AB lies on side PQ, side CD lies on side QR, and one of the remaining vertices lies on side RP. There are positive integers a, b, c, and d such that the area of triangle PQR can be expressed in the form $\frac{a+b\sqrt{c}}{d}$, where a and d are relatively prime to each other, and c is not divisible by the square of any prime. Find $a + b + c + d$.

4. Let $ABCD$ be a trapezoid with $AB \parallel CD$. Two circles with diameters BC and AD, respectively and are tangent to each other. Given that $BC = 10$ and $AD = 18$, find the perimeter of $ABCD$.

5. Let $ABCDE$ be a convex pentagon such that $AB = BC = CD = DE$, $\angle ABC = \angle BCD = 108^\circ$ and $\angle CDE = 168^\circ$. Find $\angle AEB$.

1.10.2 Area, similarity, and Ceva (part 4)

6. (Ceva's theorem) Let P be a point inside triangle ABC. Line AP intersects side BC in D, line BP intersects side CA in E, and line CP intersects side AB in F.

   (a) Express each of $\dfrac{AF}{FB}, \dfrac{BD}{DC}, \dfrac{CE}{EA}$ in terms of the areas of triangles ABP, BCP, CAP.

   (b) Show that

   $$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1 \quad \text{or} \quad AF \cdot BD \cdot CE = FB \cdot DC \cdot EA.$$

   This is Ceva's theorem.

   (c) What is the converse of Ceva's theorem? Is it true?

7. Let P be a point inside triangle ABC. Line AP intersects side BC in D, line BP intersects side CA in E, and line CP intersects side AB in F. Point R lies on line BP such that $CR \parallel AD$. Point Q lies on line CP such that $BQ \parallel AD$.

(a) Find all pairs of similar triangles in this diagram.

(b) Express each of $\frac{AF}{FB}, \frac{BD}{DC}, \frac{CE}{EA}$ in terms of AP, BR, CQ.

(c) Establish Ceva's theorem.

8. In triangle ABC, M is the midpoint of BC and DM is the perpendicular bisector of BC, where D is a point on segment AC. Given that the areas of MBD and MAD are 5 and 6 respectively, compute the area of triangle ABC.

9. Use Ceva's theorem to establish the existence of the centroid, incenter, and the orthocenter of a triangle.

10. Let points $A = (0, 0)$, $B = (1, 2)$, $C = (3, 3)$, and $D = (4, 0)$. Quadrilateral $ABCD$ is cut into equal area pieces by a line passing through A. This line intersects CD at point $\left(\frac{p}{q}, \frac{r}{s}\right)$, where these fractions are in lowest terms. What is $p + q + r + s$?

Chapter 2

Geometry Challenges

2.1 Geometry project 1: Selected medium level geometry problems from AIME

1. Let $ABCD$ be a square with side length 1. Points P, Q, R, S lie on sides AB, BC, CD, DA, respectively, with $AP = BQ = CR = DS$. Segments AQ, BR, CS, DP enclose a square of area 1/1985. Find the value of $1/BP$.

2. Circles of radius 3 and 6 are externally tangent to each other and are internally tangent to a circle of radius 9. The circle of radius 9 has a chord that is a common external tangent of the other two circles. Find the square of the length of this chord.

3. The two squares share the same center O and have sides of length 1. The overlapping region is a octagon $ABCDEFGH$ with $AB = 43/99$. The area of the octagon is m/n, where m and n are relatively prime positive integers. Find $m + n$.

4. In triangle ABC, angles A and B measures 60 degrees and 45 degrees, respectively. The bisector of angle A intersects side BC at T, and $AT = 24$. The area of triangle ABC can be written in the form $a + b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.

5. Eight spheres of radius 100 are placed on a flat surface so that each sphere is tangent to two others and their centers are the vertices of a regular octagon. A ninth sphere is placed on the flat surface so that it is tangent to each of the other eight spheres. The radius of this last sphere is $a + b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.

2.2 Challenges in geometry calculations (part 1)

1. Trapezoid $ABCD$ has parallel sides AB of length 33 and CD of length 21. The other two sides are of lengths 10 and 14. The angles at A and B are acute. What is the length of the shorter diagonal of $ABCD$?

2. Let $ABCD$ be rectangle with $AB = 1$ and $BC = 10$, and let E and F be the midpoints of sides AB and CD, respectively. Point M lies on side AD such that we can fold triangle ABM along segment BM with A lying on segment EF. Compute $\angle ABM$.

3. In triangle CAP, points X and Y lie on segments CA and AP respectively. Point Z lies on segment PX such that $YZ \parallel CA$. Segments CY and PX meet in W. Given that $CX = 1$, $XA = 3$, $AY = 2$, and $YP = 4$. Compute $XW : WZ : ZP$.

4. In the plane, point P and C lie on the same side of line AB. Point P is equidistant from A and B. Segments BP and AC meet at D. Given that $\angle APB = 2\angle ACB$, $PB = 3$, and $PD = 2$, compute $AD \cdot CD$.

5. Regular hexagon $ABCDEF$ and regular hexagon $GHIJKL$ both have side length 24. The hexagons overlap, so that G is on segment AB, B is on segment GH, K is on segment DE, and D is on segment JK. If the area of $ABCDEF$ is twice that of $GBCDKL$, compute the length of segment LF.

2.3 Geometric project 2: Tangent circles

1. If circular arcs $\widehat{AC}$ and $\widehat{BC}$ have centers at B and A, respectively, then there exists a circle tangent to both $\widehat{AC}$ and $\widehat{BC}$, and to segment AB. (This configuration looks like a section of a Gothic window.) If the length of $\widehat{BC}$ is 12, find the circumference of the circle.

2. Two circles lie outside regular hexagon $ABCDEF$. The first is tangent to side AB, and the second is tangent to side DE. Both are tangent to lines BC and FA. What is the ratio of the area of the second circle to the first circle?

3. In triangle ABC, C is a right angle and M is on side AC. A circle with radius r is centered at M, is tangent to side AB, and is tangent to side BC at C. If $AC = 5$ and $BC = 12$, compute r.

4. A circle of radius 4 and a circle of radius 9 are externally tangent. A third circle is tangent externally to the two circles and to one of their common external tangents. Find all the possible values of the radius of the third circle.

5. Chad and Jordan are having a race in a circular lake. The lake has a diameter of four kilometers and there is a circular island in the middle of the lake with a diameter of two kilometers. The island and the lake share the same center. They start at one point on the edge of the lake and finish at the diametrically opposite point. Jordan makes the trip only by swimming in the water, while Chad swims to the island, runs across it, and then continues swimming. They both take the fastest possible route and, amazingly, they tie! Chad swims at two kilometers an hour and runs at five kilometers an hour. At what speed does Jordan swim?

2.4 Challenges in geometry calculations (part 2)

1. Let A, B, C, D be four points, arranged in clockwise order, on circle ω. Segments AC and BD intersect at P. Given that $AB = 3$, $BP = 4$, $PA = 5$, $PC = 6$, find the radius of the circle ω.

2. The 8×18 rectangle $ABCD$ is cut into two congruent hexagons in such a way that the two hexagons can be repositioned without overlapping to form a square. What is the perimeter of the hexagon?

3. Let $ABCD$ be a trapezoid with $AB \parallel CD$ and $CD = 2AB = 2AD$. Suppose that $BD = 6$ and $BC = 4$, find the area of the trapezoid.

4. Circle ω, centered at I, is inscribed in quadrilateral $ABCD$. Given that $AB = 4$, $BC = 6$, $DA = 3$, $DB = 5$. Find AI.

5. In triangle ABC, $AB = 4$ and $AC = 5$. Point D lies on side AB with $AD = 3$. Let X be a point on side AC. Segments BX and CD meet in P. Compute DP/PC and BP/PX for each of the following cases.

   (a) $AX = XC$ (b) $\angle ABX = \angle XBC$

2.5 Geometry project 3: Folding, unfolding, and 3-D visions

1. The height of a cylindrical pole is 12 feet and its circumference is 2 feet. A rope is attached to a point on the circumference at the bottom of the pole. The rope is then wrapped tightly around the pole four times before it reaches a point on the top directly above the starting point at the bottom. What is the minimum number of feet in the length of the rope? Express your answer in simplest radical form.

2. A θ-degree sector of a circle of radius r is folded to a right cone by aligning the two straight sides. Compute the volume of the cones.

   (a) $\theta = 180°, r = 12$ (b) $\theta = 216°, r = 10$ (c) $\theta = 288°, r = 10$

3. Consider the paper rectangle $ABCD$. Point E lies on side AB and points F and G lie on side CD. Construct segments AG, GE, EF, FB. Given that $AG = GE = EF = FB = AE = EB = FG = 5$. One can fold this rectangular piece of paper along constructed segment to obtain a regular tetrahedron. What is the volume of the tetrahedron?

4. A cone-shaped mountain has its base on the ocean floor and has a height of 8000 feet. The top $\frac{1}{8}$ of the volume of the mountain is above water. What is the depth of the ocean at the base of the mountain, in feet?

5. A sphere of radius is inscribed in a cone with a base of 6. What is the height of the cone? What is the surface area (without the base) of the cone?