Lectures on Challenging Mathematics

UC3 Combinatorics
An Invitation to Computational Mathematics

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"Cogito ergo Sum" – "I think, therefore I am"

René Descartes (1596-1650)

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Chapter 1

Combinatorics

1.1 The first look at bijection and recursive counting

1.1.1 One-to-one correspondence

1. In order to count the elements of a certain set, we might replace the original set with another set that has the same number of elements and whose elements can be more easily counted. But the question is: How do we know these two sets have the same number of elements? This is why we need to consider a *one-to-one correspondence* between the two sets. (The more sophisticated term to described this relation is *bijection*.) The following two examples might help you to understand the meaning of the one-to-one correspondence relation.

   (a) Find the number of integers between 300 and 500 satisfy the following condition:

   It either begins with a 4 or end with a 4, but not both.

   There are many way to solve this problem. One clever way is to consider the following: There is a one-to-one correspondence between the set of integers between 300 and 399 (inclusive) that satisfy the condition and the set of integers between 400 and 500 (inclusive) that do not satisfy the condition. Do you see this one-to-one correspondence? Can you solve the problem based on this observation?

   (b) Each vertex of a cube are colored in a different color. How many equilateral triangles can be formed by using the vertices of a cube? (Two equilateral triangles are considered different if their vertices are not of the same colors.)

   A clever way of solving this problem is to set up a one-to-one correspondence between the set of equilateral triangles and the set of the vertices. Can you complete this solution now?

2. Tina randomly selects a number from the set $\{1, 2, \ldots, 10\}$, and Sergio randomly selects a number from the set $\{1, 2, \ldots, 10\}$. What is probability that Sergio's number is larger than Tina's number?

3. Alice rolls two octahedral dice with the numbers $2, 3, 4, 5, 6, 7, 8, 9$. What's the probability the two dice sum to 11?

4. How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?

5. A box contains exactly five chips, three red and two white. Chips are randomly removed one at a time without replacement until all the red chips are drawn or all the white chips are drawn. What is the probability that the last chip drawn is white?

1.1.2 Recursive counting (part 1)

6. There are n lines in a plane. Let I_n denote the maximum number of intersection points can be produced.
   (a) Establish the *recursive* relation $I_{n+1} = I_n + n$.
   (b) Establish the *explicit* formula $I_n = \frac{n(n-1)}{2}$.
   (c) Set a bijection to show that $I_n = \binom{n}{2}$.

7. (Continuation) Let r_n be the maximum number of *simple* rays can be produced. (The end point of a simple ray is an intersection point, and there is no other intersection point in the interior of the ray.)
   (a) Derive a recursive relation for r_n.
   (b) Derive an explicit formula for r_n.

8. (Continuation) Denote by s_n the maximum number of *simple* segments can be produced. (The end points of a simple segment are two intersection points, and there is no other intersection point in the interior of the segment.)
   (a) Derive a recursive relation for s_n.
   (b) Derive an explicit formula for s_n.
   (c) Determine the maximum number of simple segments on each line. Find another way to establish the explicit formula established in part (b).

9. (Continuation) Let R_n be the maximum number of regions (open and closed; that is, unbounded and bounded) in the plane formed by the lines.
   (a) Derive a recursive relation for R_n.
   (b) Derive an explicit formula for R_n.

10. (Continuation) Denote by C_n the maximum number of closed regions in the plane bounded by the lines, and denote by O_n the maximum number of open regions in the plane bounded by the lines.

(a) Derive a recursive relation for C_n.

(b) Derive an explicit formula for C_n.

(c) Derive a recursive relation for O_n.

(d) Derive an explicit formula for O_n.

(e) There are many implicit relation between r_n, s_n, R_n, C_n, O_n. Find as many as you can and justify them conceptually.

1.2 The second look at bijection and recursive counting

1.2.1 The famous model: star-and-bars or balls-and-urns

1. Three friends have a total of six identical pencils, and each one has at least one pencil. In how many ways can this happen? What if the condition of "each one has at least one pencil" is removed? What if there are seven pencils? Do you notice your answers are all binomial coefficients?

2. Let m and n be positive integers. Show that there are $\binom{n-1}{m-1}$ ordered m-tuples $(x_1, x_2, \ldots, x_m)$ of positive integers satisfying the equation $x_1 + x_2 + \cdots + x_m = n$.

   There are many approaches (or models, such as, stars-and-bars or balls-and-urns models, etc.) to solve this problem. In general, say, for $n = 10$ and $m = 4$, we consider the nine spaces between the ten 1's in the following arrangement:

   $$(1\_\_1\_\_1\_\_1\_\_1\_\_1\_\_1\_\_1\_\_1\_\_1)$$

   Define two possible states, 0 and 1, for each space. If the space is in state 0, we put a " + " there. If the space is in state 1, we put a *separator*, ") + (" there. We choose three of these spaces and put a separator in each. For example, $(x_1, x_2, x_3, x_4) = (3, 1, 2, 4)$ corresponds 001101000 which in turn corresponds to

   $$(1 + 1 + 1) + (1) + (1 + 1) + (1 + 1 + 1 + 1) = 3 + 1 + 2 + 4 = 10.$$

   Complete this approach for general n and m?

3. Show that there are $\binom{n+m-1}{m-1}$ ordered m-tuples $(x_1, x_2, \ldots, x_m)$ of nonnegative integers satisfying the equation $x_1 + x_2 + \cdots + x_m = n$.

   Establish this result in two ways:

   - One by converting m-tuples $(x_1, x_2, \ldots, x_m)$ of nonnegative integers to satisfying the equation $x_1 + x_2 + \cdots + x_m = n$ to m-tuples of $(y_1, y_2, \ldots, y_m)$ of positive integers satisfying another equation ...
   - Consider a row of $n + m - 1$ of objects! (Why $n + m - 1$?)

4. The director of student activities in a boarding school wants to distribute 61 (identical) concert tickets to five dorms – Amen Hall, McConnell Hall, Langdell Hall, Peabody Hall, Soule Hall. Each dorm shall get at least one ticket. In how many ways can this be done?

5. (Continuation) What if no dorm gets more tickets than the sum of the numbers of tickets the other dorms get?

1.2.2 Recursive counting (part 2)

6. Given a sphere, a circle is called a great circle if it is the intersection of the sphere with a plane passing through its center. Five distinct great circles dissect the sphere (the surface, not the interior) into n regions. What is the minimum value of n?

7. (Continuation) What is the maximum value of n?

8. In how many ways can a row of 10 squares be each colored either red or green or blue in such a way that no adjacent squares are in the same color?

9. For every positive integer $n \geq 3$, let a_n denote the number of way to color each vertex of a regular n-gon into red or green or blue such that no adjacent vertices are in the same color.

   (a) Find a_3 and a_4.

   (b) Explain why it is clear that a_5 is less than $3 \cdot 2^4$.

   (c) Explain why it is true that $a_5 + a_4 = 3 \cdot 2^4$.

   (d) Evaluate a_6 and a_7.

10. In how many ways can a row of 12 squares be each colored either red or green in such a way that no two red squares are adjacent?

1.3 The third look at bijection and recursive counting

1.3.1 Bijections in geometric models

1. Imagine 173 unit squares arranged in a row. If a rectangle consists of a single square, or a combination of consecutive squares, compute the number of rectangles that can be formed.

2. There are h horizontal lines and v vertical lines drawn in a plane. There are r ways to choose four lines such that a rectangular region is enclosed. Given that r is divisible by 1001. Find the minimum value of $h + v$.

3. Ten points are selected on the positive x-axis, $\mathbf{X}^+$, and five points are selected on the positive y-axis, $\mathbf{Y}^+$. The fifty segments connecting the ten points on $\mathbf{X}^+$ to the five points on $\mathbf{Y}^+$ are drawn. What is the maximum possible number of points of intersection of these fifty segments in the interior of the first quadrant?

4. Let n be a positive integer. Points $A_1, A_2, \ldots, A_n$ lie on a circle. For $1 \le i < j \le n$, we construct segment A_iA_j. Let S denote the set of all such segments. Determine the maximum number of intersection points that can produced by the elements in S.

5. Nine different points $C, O, M, P, U, T, E, R, S$ are randomly chosen on a circle. What is the probability that triangles COP and SET intersect?

1.3.2 Mixed exercises

6. On a regular ruler of length 20 (with 21 evenly spaced marks, namely, 0, 1, 2, ..., 20), all but six marks can be erased. Three of the remaining marks are 0, 1, 2. If ones wants to be able to measure all integer distances up to 13, what could be the three other unerased marks? Can we achieve this goal by erasing all but five marks?

7. Let n be a positive integer. In how many ways can one write a sum of (at least two) positive integers that add up to n? Consider the same set of integers written in a different order as being different. (For example, there are 3 ways to express 3 as $3 = 1 + 1 + 1 = 2 + 1 = 1 + 2$.)

8. Determine the number of 5-tuples (a, b, c, d, e) of integers with $1 \le a \le b \le c \le d \le e \le 20$.

   You might wish that the condition is $1 \le a < b < c < d < e \le 20$, aren't you? Well, in this case, for one approach to solve the problem, this wishful thinking might not be a bad start. Considering a few numbers summing up to 20 leads to another approach. Complete both approaches!

9. Given eight distinguishable rings, find the number of possible five-ring arrangements on the four fingers (not the thumb) of one hand. (The order of the rings on each finger is significant, but it is not required that each finger have a ring.)

10. In Zuminglish, all words consist only of the letters `M`, `O`, `P`. As in English, `O` is said to be a vowel and `M` and `P` are consonants. A string of `M`s, `O`s, and `P`s is a word in Zuminglish if and only if between any two `O`s there appear at least two consonants. Let N denote the number of 10-letter Zuminglish words. Determine the remainder obtained when N is divided by 1000.

1.4 Entry level Combinatorics problems from the first 10 years of AIME

1. An ordered pair (m, n) of non-negative integers is called *simple* if the addition of $m+n$ in base 10 requires no carrying. Find the number of simple ordered pairs of non-negative integers that sum to 1492.

2. One commercially available ten-button lock may be opened by depressing – in any order – the correct five buttons. Suppose that these locks are redesigned so that sets of as many as nine buttons or as few as one button could serve as combinations. How many additional combinations would this allow?

3. Twelve evenly spaced points are marked on a circle. Ted randomly is choosing a polygon with three or more sides with some (or all) of the twelve points as vertices. What is the probability that Ted is going to end up with a regular polygon? (Polygons are distinct unless they have exactly the same vertices.)

4. The increasing sequence $2, 3, 5, 6, 7, 10, 11, \ldots$ consists of all positive integers that are neither the square nor the cube of a positive integer. Find the 500^{th} term of this sequence.

5. A positive integer is called *ascending* if, in its decimal representation, there are at least two digits and each digit is less than any digit to its right. How many ascending positive integers are there?

6. Expanding $(1+0.2)^{1000}$ by the binomial theorem and doing no further manipulation gives

$$\binom{1000}{0}(0.2)^0+\binom{1000}{1}(0.2)^1+\binom{1000}{2}(0.2)^2+\cdots+\binom{1000}{1000}(0.2)^{1000} = A_0+A_1+\cdots+A_{1000},$$

where $A_k = \binom{1000}{k}(0.2)^k$ for $k = 0, 1, 2, \ldots, 1000$. For which k is A_k largest?

7. In Pascal's triangle, each entry is the sum of the two entries above it. The first few rows of the triangle are shown below:

Row 0						1					
Row 1					1		1				
Row 2				1		2		1			
Row 3			1		3		3		1		
Row 4		1		4		6		4		1	
Row 5	1		5		10		10		5		1

In which row of Pascal's triangle do three consecutive entries occur that are in the ratio of $3 : 4 : 5$?

8. When a certain biased coin is flipped 5 times, the probability of getting heads exactly once is not equal to 0 and is the same as that of getting heads exactly twice. Let i/j, in lowest terms, be the probability that the coin comes up heads exactly 3 times out of 5. Find $i + j$.

9. Say that a rational number is special if its decimal expansion is of the form $0.\overline{abcdef}$, where a, b, c, d, e, and f are (not necessarily distinct) digits that include each of the digits 2, 0, 1, and 5 at least once (in some order). How many special rational numbers are there?

10. Starting at $(0, 0)$, an object moves in the coordinate plane via a sequence of steps, each of length one. Each step is left, right, up, or down, all four being equally likely. What is the probability that the object reaches $(2, 2)$ in six or fewer steps?

1.5 Let's count (part 1)

1. Austin currently owns some shirts, pants, and pairs of shoes; he chooses one of each to create an outfit. If he were to obtain one more shirt, his total number of outfits would increase by 48. Similarly, if he bought another pair of pants he would have 90 more outfits, while an extra pair of shoes would result in 120 more outfits. How many outfits can he currently create?

2. In the game of Minesweeper, a number on a square denotes the number of mines that share at least one vertex with that square. A square with a number may not have a mine, and the blank squares are undetermined. How many ways can the mines be placed in this configuration?

	2		1		2

3. A classroom has 5 rows and 7 columns of desks. There is one student at each of the 35 desks. The teacher wishes to reassign the seats in the following way: one *special* student will be chosen to stay put, and each other student moves either one seat to the left or right, or one seat forward or backward. In how many ways can the special student be chosen?

4. In ARMLvania, license plates use only the digits $1, 2, \ldots, 9$, and each license plate contains exactly 8 digits. On each plate, all digits are distinct, and for all $k \leq 8$, the k^{th} leftmost digit is at least k. Compute the number of valid ARMLvanian license plates.

5. A house with a square floor plan is to have each of its four outer walls finished with brick, stone, stucco or wood. If no two adjacent outer walls are to have the same finish, in how many ways can the house's walls be constructed? (One such way is brick on the front and back and stone on the left and right.)

6. How many 19-digit binary sequences have more 1s than 0s? How many 20-digit binary sequences have at least as many 1s as 0s?

7. The nine delegates to the Economic Cooperation Conference include 2 officials from Mexico, 3 officials from Canada, and 4 officials from the United States. During the opening session, three of the delegates fall asleep. Assuming that the three sleepers were determined randomly, what is the probability that exactly two of the sleepers are from the same country?

8. A fair die is rolled four times. What is the probability that each of the final three rolls is at least as large as the roll preceding it?

9. What is the average (mean) of all 5-digit numbers that can be formed by using each of the digits 1, 3, 5, 7, and 8 exactly once?

10. A computer program is a function that takes in 4 bits, where each bit is either a 0 or a 1, and outputs TRUE or FALSE. How many computer programs are there?

1.6 Let's count (part 2)

1. A classroom has 3 rows and 3 columns of desks. The teacher wishes to reassign the seats in such a way that each student moves either one seat to the left or right, or one seat forward or backward. Each student moves only once. Suppose initially the central desk is empty and there is one student at each of the other desks. In how many ways can this be done, if the teacher moves the students sequentially, that is, at each step, the teacher moves one of the students to a currently empty desk? What if the empty desk is one in the corner of the classroom? What if the empty desk is neither the central desk nor the corner one?

2. (Continuation) Suppose initially the central desk is empty. Find the number of ways to reassign seats if the teacher moves the students all at once, that is, they all stand up and change their seats in such a way that each student moves either one seat to the left or right, or one seat forward or backward.

3. A binary palindrome is a positive integer whose standard base 2 (binary) representation is a palindrome (reads the same backward or forward). (Leading zeroes are not permitted in the standard representation.) For example, 2015 is a binary palindrome, because in base 2 it is 11111011111. How many positive integers less than 2015 are binary palindromes?

4. For how many three-element sets of positive integers $\{a, b, c\}$ is it true that the product abc is equal to 210?

5. (Continuation) What if the product is equal to 2310?

6. Each face of a cube is painted either red or blue, each with probability 1/2. The color of each face is determined independently. What is the probability that the painted cube can be placed on a horizontal surface so that the four vertical faces are all the same color?

7. Two unit squares are selected at random without replacement from an $n \times n$ grid of unit squares. Find the least positive integer n such that the probability that the two selected unit squares are horizontally or vertically adjacent is less than $\frac{1}{2015}$.

8. For each face of a cube, Bob picks a random edge on that face and draws an arrow from the midpoint of that edge to the midpoint of the opposing edge on that face. If Bob picks the edge independently for each face, what is the probability that Allie the ant can crawl in a loop around the cube by following some of the arrows?

9. Three distinct vertices of a cube are chosen at random. What is the probability that the plane determined by these three vertices contains points inside the cube?

10. In a drawer Sandy has 7 pairs of socks, each pair a different color. On Monday Sandy selects two individual socks at random from the 14 socks in the drawer. On Tuesday Sandy selects two of the remaining 12 socks at random and on Wednesday two of the remaining 10 socks at random. What is probability that Wednesday is the first day Sandy selects matching socks?

1.7 Practice with binomial coefficients (part 1)

1. The PEA mathematics department is to hold a meeting to discuss pedagogy. After a long conversation among 23 members of the department, they decide to split into 5 groups of three and 2 groups of four to continue their discussion. In how many ways can this be done?

2. In the expansion of $(ax+b)^{2000}$, where a and b are relatively prime positive integers, the coefficients of x^2 and x^3 are equal. Find $a+b$.

3. How many ways are there to place two `A`'s, two `B`'s, two `C`'s, and two `D`'s in four distinguishable boxes such that every box has two letters?

4. What is the value of the constant term in the expansion of $\left(\left(x+\frac{1}{x}\right)^2-4\right)^{20}$?

5. There are 10 people who want to choose a committee of 5 people among them. They do this by first electing a set of 1, 2, 3, or 4 committee leaders, who then choose among the remaining people to complete the 5-person committee. In how many ways can the committee be formed, assuming that people are distinguishable? (Two committees that have the same members but different sets of leaders are considered to be distinct.)

6. A coin is altered so that the probability that it lands on heads is less than $\frac{1}{2}$ and when the coin is flipped four times, the probability of an equal number of heads and tails is $\frac{1}{6}$. What is the probability that the coin lands on heads?

7. Let $A=\{a_1,a_2,\ldots,a_{100}\}$ and $B=\{1,2,\ldots,50\}$. Determine the number of surjective functions f from A to B such that $f(a_1)\le f(a_2)\le\cdots\le f(a_{100})$. What if f does not need to be surjective?

8. Given that
$$\frac{1}{2!17!}+\frac{1}{3!16!}+\frac{1}{4!15!}+\frac{1}{5!14!}+\frac{1}{6!13!}+\frac{1}{7!12!}+\frac{1}{8!11!}+\frac{1}{9!10!}=\frac{N}{1!18!}.$$
find the greatest integer that is less than $N/100$.

9. The expression $(x+y+z)^{2006}+(x-y-z)^{2006}$ is simplified by expanding it and combining like terms. How many terms are in the simplified expression?

10. For $k\ge 3$, we define an ordered k-tuple of real numbers $(x_1,x_2,\ldots,x_k)$ to be *special* if, for every i such that $1\le i\le k$, the product $x_1x_2\cdots x_k=x_i^2$. Compute the smallest value of k such that there are at least 2009 distinct special k-tuples.

1.8 Let's count (part 3)

1. Farmer John has 5 cows, 4 pigs, and 7 horses. How many ways can he pair up the animals so that every pair consists of animals of different species? (Assume that all animals are distinguishable from each other.)

2. There are 10! permutations $(s_0, s_1, \ldots, s_9)$ of $(0, 1, \ldots, 9)$. How many of them satisfy $s_k \geq k-2$ for $k = 0, 1, \ldots, 9$?

3. A bag contains 20 lavender marbles, 12 emerald marbles, and some number of orange marbles. If the probability of drawing an orange marble in one try is $\frac{1}{y}$, compute the sum of all possible integer values of y.

4. Determine the number of ways to write $1, 2, \ldots, 12$ evenly around a circle such that the sum of every three adjacent numbers is divisible by 3. (Two schemes are considered the same if one can be obtained by rotating the other.)

5. David is in a math class with other 14 students. The classroom has 15 seats for these 15 students. Everyday, the 15 students line up in random order and then enter the classrooms one by one. Except David, each student seats himself randomly. David, however, prefers the seat next to the door and choose it is unoccupied when he enters the classroom. What is the probability that David can sit in his favorite seat?

6. Three points are chosen from the vertices of the convex octagon. What is the probability that they form a triangle with none of its sides being a side of the octagon.

7. Twelve identical red cards and seven identical black cards are laid in a row on a table. How many different arrangements are possible if no two black cards are allowed to be adjacent to each other?

8. Four standard, six-sided dice are to be rolled. If the product of their values turns out to be an even number, what is the probability their sum is odd?

9. Determine the number of integer arithmetic sequences (a_1, a_2, a_3, a_4) with $1 \leq a_1 < a_2 < a_3 < a_4 \leq 2012$.

10. A given sequence $r_1, r_2, \ldots, r_n$ of distinct real numbers can be put in ascending order by means of one or more "bubble passes." A bubble pass through a given sequence consists in comparing the second term with the first term and exchanging them if and only if the second term is smaller, then comparing the third term with the current second term and exchanging them if and only if the third term is smaller, and so on, in order, through comparing the last term, r_n, with its current predecessor and exchanging them if and only if the last term is smaller.

$\underline{1}$	$\underline{9}$	8	7
1	$\underline{9}$	$\underline{8}$	7
1	8	$\underline{9}$	$\underline{7}$
1	8	7	9

The figure on the right hand side above shows how the sequence $1, 9, 8, 7$ is transformed into the sequence $1, 8, 7, 9$ by one bubble pass. The numbers compared at each step are underlined.

Suppose that $n = 40$, and that the terms of the initial sequence $r_1, r_2, \dots, r_{40}$ are distinct from one another and are in random order. Let p/q, in lowest terms, be the probability that the number that begins as r_{20} will end up, after one bubble pass, in the 30^{th} place. Find $p + q$.

1.9 Practice with binomial coefficients (part 2)

1. Two identical decks of cards are shuffled together. What is the probability that the first 52 cards will contain all eight Aces?

2. Consider a 3×3 point array. Five of the nine points will be colored in red, and the rest will be colored in blue.

   (a) Determine the total number of coloring schemes. (Two coloring schemes are considered different even though one can be obtained from the other via a rotation and/or a reflection.)

   (b) Some these coloring schemes satisfy the following property: There exist three blue points that are collinear and there does not exist three red points that are collinear. How many such coloring schemes are there?

3. Find the smallest positive integer n for which the expansion of $(xy - 3x + 7y - 21)^n$, after like terms have been collected, has at least 1996 terms.

4. Boxes of height 8 inches, 9 inches, and 10 inches are available. There are an ample supply of boxes in each of three sizes. Every possible stack of six boxes with a unique stacking sequence is to be made once, such that each stack is one box wide and one box deep. If John is to select one of these stacks at random, what is the probability his stack is exactly 54 inches?

5. A license plate consists of 8 digits. It is called *even* if it contains an even number of 0's. Find the number of even license plates.

6. For how many three-element sets of positive integers $\{a, b, c\}$ is it true that the product abc is equal to 30030.

7. A 5-tuple $(x_1, x_2, x_3, x_4, x_5)$ of nonnegative integers is called *magical* if exactly one of the 5 numbers is odd and the sum of these 5 five numbers is equal to 99. How many magical 5-tuples are there?

8. Evaluate $\frac{\binom{11}{0}}{1} + \frac{\binom{11}{1}}{2} + \frac{\binom{11}{2}}{3} + \cdots + \frac{\binom{11}{11}}{12}$.

9. The expression $(x+y-z)^{100}$ is expanded and simplified. What is the sum of all the coefficients? What is the sum of all the positive coefficients? How many terms are there altogether?

10. Show that

$$\sum_{k=0}^{9} \frac{(k+1)\cdot(38-k)!}{29!(9-k)!} = \frac{40!}{31!9!},$$

(By the way, this problem is closely related to a problem we did in the previous section. Which problem is that, and how do these problems relate to each other?)

1.10 Let's count (part 4)

1. How many subsets A of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ have the property that no two elements of A sum to 11?

2. When a four-digit number with units digit 2 is divided by a one-digit number, it has a remainder of 1. How many such four-digit numbers are there?

3. Say that an ordered triple (a, b, c) is *pleasing* if

   (a) a, b, and c are in the set $\{1, 2, \ldots, 17\}$;

   (b) both $b - a$ and $c - b$ are greater than 3;

   (c) at least one the numbers $b - a$ and $c - b$ is equal to 4.

   How many pleasing triples are there? How many triples are there if we remove condition (c)?

4. Twelve chairs are set up in a row for the Princeton garlic-eating contest. Only five eaters attend the competition, but none will sit next to any other. In how many ways can the eaters be seated? What if there are twenty chairs and eight eaters?

5. An integer is called *snakelike* if its decimal representation $a_1a_2a_3 \ldots a_k$ satisfies $a_i < a_{i+1}$ if i is odd and $a_i > a_{i+1}$ if i is even. How many snakelike integers between 1000 and 9999 have four distinct digits?

6. Twelve fair dice are rolled. What is the probability that the product of the numbers on the top faces is prime?

7. Boston Yankees and New York Red Sox play a series. The first team to win three games wins the series. Each team is equally likely to win each game, there are no ties, and the outcomes of the individual games are independent. If Boston Yankees wins the second game and New York Red Sox wins the series, what is the probability that Boston Yankees wins the first game?

8. How many positive integers less than or equal to 240 can be expressed as a sum of distinct factorials? Consider 0! and 1! to be distinct.

9. When rolling a certain unfair six-sided die with faces numbered 1, 2, 3, 4, 5 and 6, the probability of obtaining face F is greater than $1/6$, the probability of obtaining the face opposite F is less than $1/6$, the probability of obtaining each of the other face is $1/6$, and the sum of the numbers on each pair of opposite faces is 7. When two such dice are rolled, the probability of obtaining a sum of 7 is $47/288$. Given that the probability of obtain face F is m/n, where m and n are relatively prime positive integers, find $m + n$.

10. A frog is placed at the origin on the number line, and moves according to the following rule: in a given move, the frog advances to either the closest point with a greater integer coordinate that is a multiple of 3, or to the closest point with a greater integer coordinate that is a

multiple of 13. A *move sequence* is a sequence of coordinates which corresponds to valid moves, beginning with 0, and ending with 39. For example, 0, 3, 6, 13, 15, 26, 39 is a move sequence. How many move sequences are possible for the frog?