Lectures on Challenging Mathematics

UC3 Algebra
An Invitation to Computational Mathematics

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"Cogito ergo Sum" – "I think, therefore I am"

René Descartes (1596-1650)

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Chapter 1

Algebra Knowledge

1.1 More on distance and motion

1. A certain job can be finished either (1) by Fat works on it for 6 days and then Aft works on it for another 12 days; or (2) by Fat works on it for 8 days and then Aft works on it for another 6 days. If Fat works on it for 3 days, how long does it take Aft to finish the left over job?

2. Given a regular clock, determine when between 4:00 pm and 5:00 pm the minute hand and the hour hand are perpendicular to each other.

3. Michael walks at the rate of 5 feet per second on a long straight path. Trash pails are located every 200 feet along the path. A garbage truck travels at 10 feet per second in the same direction as Michael and stops for 30 seconds at each pail. As Michael passes a pail, he noticed the truck ahead of him just leaving the next pail. How many times will Michael and the truck meet?

4. The workers in a factory produce widgets and whoosits. For each product, production time is constant and identical for all workers, but not necessarily equal for the two products. In one hour, 100 workers can produce 300 widgets and 200 whoosits. In two hours, 60 workers can produce 240 widgets and 300 whoosits. In three hours, 50 workers can produce 150 widgets and m whoosits. Find m.

5. Walking inside the subway station at 34^{th} street, coach Fat observed that every 11 minutes there was an A-train passing by going up town, and every 10 minutes there was an A-train coming by going down town. If coach Fat maintained constant speed, what is the time gap between two consecutive A-trains going the same direction?

1.2 Revisit the quadratic function and its graph (part 1)

1. Find an equation of the following parabolas.
   (a) It has vertex at $(3, -4)$ and passes through the point $(1, 2)$.
   (b) It has x-intercepts $(4, 0)$ and $(-6, 0)$ and passes through the point $(2, 3)$.

2. For every $n = 1, 2, \ldots, 2008$, let d_n denote the length of the segment joining the x-intercepts of the graph of $y = (n^2 + n)x^2 - (2n + 1)x + 1$. Compute $d_1 + d_2 + \cdots + d_{2008}$.

3. Let f be a function for which $f\left(\frac{x}{3}\right) = x^2 + x + 1$. Find the sum of all values of z for which $f(3z) = 7$.

4. Consider $y = f(x) = ax^2 - bx + c$ with $f(-1) > 0$, $f(0) < 0$, and $f(1) = 0$. Make a careful sketch a graph of $y = f(x)$, and then make careful sketches, in the same coordinate system, of the graphs of $y = ax^2 + bx + c$ and $y = -ax^2 + bx - c$. Identify the relations between the three sketches.

5. (Continuation) Determine if a, b, c, $a + b + c$, $a - b + c$, $b^2 - 4ac$ are always positive, zero, or negative.

1.3 Bouncing balls and the domain and the range of the infinite geometric series (part 1)

(Most of the problems in this section are selected from PEA Math 3 materials, with Richard Parris as the main author.)

1. A speckled green superball has a 75% rebound ratio. When you drop it from a height of 16 feet, it bounces and bounces and bounces ....
   (a) How high does the ball bounce after it strikes the ground for the third time?
   (b) How high does the ball bounce after it strikes the ground for the seventeenth time?
   (c) When it strikes the ground for the second time, the ball has traveled a total of 28 feet in a *downward* direction. Verify this. How far downward has the ball traveled when it strikes the ground for the seventeenth time?

2. (Continuation) At the top of its second rebound, the ball has traveled 21 feet upward.
   (a) At the top of its seventeenth rebound, how far upward has the ball traveled?
   (b) At the top of its seventeenth rebound, how far has the ball traveled in total?

3. (Continuation) How far would the ball travel if you just let it bounce and bounce and bounce ....

4. When $t = 4/5$, the infinite series $1 + t + t^2 + t^3 + \cdots$ *equals* or *converges* to 5. When $t = -2/3$, the series converges to 3/5. What do these statements mean? (Consider the partial sums $s_0 = 1$, $s_1 = 1 + t$, $s_2 = 1 + t + t^2$, ..., $s_n = 1 + t + \cdots + t^n$.)

5. (Continuation) For what t-values is it correct to say that the series $1 + t + t^2 + t^3 + \cdots$ has a *sum*? What is this sum?

1.4 Revisit the quadratic function and its graph (part 2)

1. After rolling off the end of a horizontal ramp, a ball follows a curved trajectory to the floor. To test a theory that says that the trajectory can be described by an equation $y = h - ax^2$, Sasha makes some measurements. The end of the ramp is 128 cm above the floor, and the ball lands 80 cm downrange. In order to catch the ball in mid-flight with a cup that is 78 cm above the floor, where should Sasha place the cup?

2. A parabola with equation $y = ax^2 + bx + c$ is reflected about the x-axis. The parabola and its reflection are translated horizontally five units in opposite directions to become the graphs of $y = f(x)$ and $y = g(x)$, respectively. Which of the following describes the graph of $y = f(x) + g(x)$?

   (a) a parabola tangent to the x-axis

   (b) a parabola not tangent to the x-axis

   (c) a horizontal line

   (d) a non-horizontal line

   (e) the graph of a cubic function

3. Plot points $(-2, 1)$, $(3, 1)$, and $(0, 7)$. There is a quadratic function whose graph passes through these three points. Sketch the graph. Find its equation in two ways.

   First, begin with the equation $y = ax^2 + bx + c$ and use the three points to find the values of a, b, and c. (One of these values is essentially given to you. Which one, and why?)

   Second, begin with the equation $y = a(x - h)^2 + k$ and use the three points to determine a, h, and k. (One of these values is almost given to you. Which one, and why?)

   Your two equations do not look alike, but they should be equivalent. Check that they are.

4. Find the values of m such that $7x^2 - (m + 13)x + m^2 - m - 2 = 0$ has one root between 0 and 1 and one root between 1 and 2.

5. Find the values of a and b if the solution set for the inequality $ax^2 + abx + b > 0$ is $1 < x < 2$.

1.5 Computations with logarithm (part 1)

1. A circle has a radius of $\log(a^2)$ and a circumference of $\log(b^4)$, where a and b are positive real numbers. What is $\log_a b$?

2. Let a and b be rational numbers with $a > 0$ such that $10^{30\log 4+50\log 6-70\log 8} = a^b$. Determine the maximum value of $a+b$.

3. What is the value of the expression

$$\frac{1}{\log_2 100!}+\frac{1}{\log_3 100!}+\frac{1}{\log_4 100!}+\cdots+\frac{1}{\log_{100} 100!}.$$

4. Given that $\log 2 = a$ and $\log 3 = b$. Express $\log_5 12$ in terms of a and b.

5. Compute all real values of x such that $\log_2(\log_2 x) = \log_4(\log_4 x)$.

1.6 Focus, directrix, tangents, and parabola (part 1)

(Most of the problems in this section are selected from PEA Math 2 materials, with Richard Parris as the main author.)

1. Find an equation that says that $P = (x, y)$ is equidistant from $F = (2, 0)$ and the y-axis. Plot four points that fit this equation. The configuration of all such points P is called a *parabola*.

2. Find an equation of all the points that are equidistant to point $F = (4, 2)$ and line $\ell : y = 4$.

3. Let $F = (3, 3)$. The points $P = (x, y)$ that are equidistant from F and the line $y = -5$ form a parabola.

   (a) Find the points where this parabola meets the x-axis.

   (b) Find the coordinates of the vertex.

   (c) Find an equation for the parabola.

4. Given a line λ (*Greek "lambda"*) and a point F not on λ, let P be on the parabola of all points that are equidistant from F and from λ. Let N be the point on λ closest to P. Explain why any other point Q on the parabola must be closer to F than to N. What does this tell you about the perpendicular bisector of FN?

5. Find three specific points that are equidistant from $F = (4, 0)$ and the line $y = x$.

1.7 Sums and products (part 1)

1. Factor $a^2+b^2-c^2+2ab$ and $a^2+b^2-c^2-2ab$, and expand $(a+b+c)(a+b-c)(b+c-a)(c+a-b)$

2. Joseph and Tim are trying to find a polynomial $p(x)$ with integer coefficients such that $x = 2+\sqrt{3}$ is a root of $p(x)$. Joseph started with $x^2 = (2+\sqrt{3})^2$. Tim started with $x-2=\sqrt{3}$. Complete both approaches.

3. Find a polynomial $p(x)$ with integer coefficients with $\sqrt{3}+\sqrt{7}$ as one of its roots.

4. It is often convenient to use what is called *sigma notation* to describe a series. For example,

$$\sum_{k=0}^{20} 3000(1.05)^k = 3000 + 3150 + \cdots + 3000(1.05)^{20}$$

and

$$18 - 12 + 8 - \cdots + 18(-2/3)^{49} = \sum_{n=1}^{50} 18\left(-\frac{2}{3}\right)^{n-1} \quad \text{or} \quad \sum_{p=0}^{49} 18\left(-\frac{2}{3}\right)^{p}$$

The symbol Σ is the Greek letter *sigma*, which corresponds to the English S (for *sum*). Identify the three key components of the sigma notation. (This will help you to operate your calculator that has the built-in capacity to evaluate sigma notation.)

Express (a), (b), (c), (d) using sigma notation, and then evaluate the series (e), (f), (g), (h).

(a) $1+4+9+\cdots+361$

(b) $8+4+2+\cdots+\frac{1}{4}+\frac{1}{8}$

(c) $1+3+7+15+31+63+127+255$

(d) $\log 1 + \log \frac{1}{2} + \cdots + \log \frac{1}{10}$

(e) $\sum_{n=1}^{100} 5n - \sum_{n=7}^{95} 5(n-7)$

(f) $\sum_{n=1}^{1000} (-1)^n n$

(g) $\sum_{n=1}^{10} (2n-1) + \sum_{n=1}^{10} (2^n - 1)$

(h) $\sum_{n=-10}^{10} \log 2^n$

5. Sasha builds a sugar-cube pyramid by stacking centered square layers. The dimension of each layer is one less than the dimension of the layer immediately below it. The bottom layer is n-by-n. Sasha would like a formula for the total number of sugar cubes in such a pyramid. Sasha knows the formula $\frac{1}{2}n(n+1)$ for the sum of consecutive integers $1+2+3+\cdots+n$. Because $\frac{1}{2}n(n+1)$ is a quadratic function of n (it can be written in the form an^2+bn+c), Sasha guesses that the formula for $S(n) = 1+4+9+\cdots+n^2$ is a cubic function of n. In other words, $S(n)$ can be written an^3+bn^2+cn+d. Use the data: $S(1)$, $S(2)$, $S(3)$, and $S(4)$ to determine values for a, b, c, and d. Test your formula on $S(5)$.

   Use the formula to express the volume of the sugar-cube pyramid as a fractional part of the volume of an n-by-n-by-n cube. What is this ratio when n is a very large number?

1.8 Focus, directrix, tangents, and parabola (part 2)

(Most of the problems in this section are selected from PEA Math 2 materials, with Richard Parris as the main author.)

1. Write an equation that says that $P = (x, y)$ is on the parabola whose focus is $(2, 1)$ and whose directrix is the line $x + 1 = 0$.

2. Let λ be the line $y = 1$ and F be the point $(-1, 2)$. Verify that the point $(2, 6)$ is equidistant from λ and F. Sketch the configuration of *all* points P that are equidistant from F and λ. Recall that this curve is called a *parabola*. Point F is called its *focus*, and line λ is called its *directrix*. Find an equation that says that $P = (x, y)$ is on the parabola.

3. (Continuation) Let $N = (2, 1)$, and find an equation for the perpendicular bisector of FN. As a check, verify that $P = (2, 6)$ is on this line. (Why could this have been predicted?) Explain why this line intersects the parabola only at P.

4. Graph the curve $4y = (x + 2)(x - 6)$, and mark the point $F = (2, -3)$. Choose any point on the curve, and show that it is equidistant from F and the line $y = -5$. Do the same for three more points on the curve. These results suggest that the curve is the parabola whose focus is F and whose directrix is the line $y = -5$. What additional work is needed to prove this statement?

5. Show that the graph of the quadratic equation $y = x^2$ is a parabola, by finding coordinates for its focus and an equation for its directrix.

1.9 Bouncing balls and the domain and the range of the infinite geometric series (part 2)

(Most of the problems in this section are selected from PEA Math 3 materials, with Richard Parris as the main author.)

1. For the first 31 days of your new job, your boss offers you two salary options. The first option pays you \$1000 on the first day, \$2000 on the second day, \$3000 on the third day, and so on — in other words, \$1000n on the n^{th} day. The second option pays you one penny on the first day, two pennies on the second day, four pennies on the third day — the amount doubling from one day to the next. Which option do you prefer, and why?

   You have chosen the second payment option, and on the thirty-first day your boss pays you the wages for that day — in pennies. You wonder whether all these coins are going to fit into your dormitory room, which measures 12 feet by 15 feet by 8 feet. Verify that a penny is 0.75 inch in diameter, and that seventeen of them make a stack that is one inch tall. Use this information to decide whether the pennies will all fit.

2. A fact from physics: The time required to fall from a height of h feet (or to rise to that height after a bounce) is $\sqrt{h}/4$ seconds. Suppose that a ball, whose rebound ratio is 64 percent, is dropped from a height of 25 feet.

   (a) When the ball strikes the ground for the second time, it will have traveled 57 feet in total. Confirm that this is a true statement.

   (b) How much time passes between the initial drop and the second impact?

   (c) Avery responds to the preceding question, "That's easy, you just divide $\sqrt{57}$ by 4 and get 1.89 seconds." Brooks responds, "I think you meant to divide $\sqrt{25}+\sqrt{32}$ by 4 and get 2.66 seconds." What do you think of these remarks, and why?

3. (Continuation) How much time passes between the initial drop and the hundredth impact? How far has the ball traveled by then?

4. (Continuation) How far does the ball travel if it is left to bounce "forever"? How much time does all this bouncing actually take?

5. For what t-values is the sum of series $1+t+t^2+t^3+\cdots$ equal to

   (a) $\dfrac{2}{3}$? (b) $\dfrac{3}{2}$? (c) 2016? (d) $\dfrac{1}{2016}$?

1.10 The discriminant of the quadratic function

1. There are two values of a for which the equation $4x^2 + ax + 8x + 9 = 0$ has only one solution for x. What is the sum of those values of a?

2. Find the values of a and b for which $x^2 + 2(1 + a)x + (3a^2 + 4ab + 4b^2 + 2) = 0$ has real solutions for x.

3. Given that a, b, and c are the lengths of the three sides of a triangle, show that the equation $b^2x^2 + (b^2 + c^2 - a^2)x + c^2 = 0$ has no real solution.

4. Determine all the points P in the coordinate plane such that there are exactly two circles passing through P and tangent to both axes.

5. Find the range of
$$\frac{x^2 + 2x}{2x^2 + 1}.$$

1.11 Fractal and recursive relation (part 1)

(Some of the problems in this section are selected from PEA Math 3 materials, with Richard Parris as the main author.)

1. Find recursions that describe the sequences $1, 1, 1, 3, 5, 9, 17, 31, \ldots$.

2. The first term of a sequence is 2005. Each succeeding term is the sum of the cubes of the digits of the previous term. What is the 2005^{th} term of the sequence?

3. In 1904 Helge von Koch invented his *snowflake*, which is probably the first published example of a *fractal*. It is the result of an endless sequence of stages: Stage 0 (the initial configuration) consists of an equilateral triangle, whose sides are 1 unit long. Stage 1 is obtained from stage 0 by replacing the *middle third* of each edge by a pair of segments, arranged so that a small equilateral triangle protrudes from that edge. In general, each stage is a polygon that is obtained by applying the middle-third construction to *every* edge of the preceding stage.

   (a) Make your own sketch of stages 0, 1, 2, and 3.

   (b) Stage 0 has three edges, and stage 1 has twelve. How many edges do stages 2 and 3 have? How many edges does stage n have?

   (c) Stage 1 has twelve vertices. How many vertices does stage n have?

   (d) How long is each edge of stage 1? of stage 2? of stage n?

   (e) What is the perimeter of stage 1? of stage 2? of stage n?

   (f) Does the snowflake have finite perimeter? Explain.

   (g) Is the area enclosed by the snowflake finite? Explain. (Try to do so without any computation.)

4. (Continuation) *The Koch snowflake* is an example of a fractal curve of infinite length. However, the area enclosed by this curve is finite. Suppose that the area enclosed by stage 0 (the initial equilateral triangle) is 1. What is the area enclosed by stage 1? by stage 2? by stage n? Show that the area enclosed by the completed snowflake can be obtained with the help of a geometric series.

5. Let $f_1(x) = \dfrac{2}{3} - \dfrac{3}{3x+1}$, and for $n \geq 2$, define $f_n(x) = f_1\left(f_{n-1}(x)\right)$.

   (a) The value of x that satisfies $f_{1001}(x) = x - 3$ can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

   (b) Consider the sequence $x_0, x_1, x_2, \ldots$, where $x_n = f_n(x_0)$ for every positive integer n. Write an recursive relation for x_{n+1} in terms of x_n. Find the relation between this recursive relation with the recursive relation $a_{n+1} = 1 - \dfrac{1}{a_n}$.

1.12 Focus, directrix, tangents, and parabola (part 3)

(Most of the problems in this section are selected from PEA Math 2 materials, with Richard Parris as the main author.)

1. When asked to find the focus and directrix for the parabola $8y = (x-4)^2 + 2$, Jamie commented, "I would rather work with $8y = x^2$ and move things around a bit." What does Jamie mean? Complete the solution under Jamie's suggestion.

2. Find the focus and the directrix for the parabola $4py = x^2$. Make a sketch that shows the meaning of the *parameter* p.

3. Draw a line λ in your notebook, and mark a point F approximately an inch away from λ. Sketch the parabola that has λ as its directrix and F as its focus. Locate the point V on the parabola, called the vertex, which is closest to the focus. Draw the line through F that is perpendicular to λ. How is this line related to V and to the parabola?

4. Find a point on the line $y = x$ that lies on the parabola whose focus is $(0, 2)$ and directrix is the x-axis. Describe the relationship between the line $y = x$ and the parabola.

5. Does $(1, 11)$ lie on the parabola defined by the focus $(0, 4)$ and the directrix $y = x$? Justify your answer.

1.13 Computations with logarithm (part 2)

1. Suppose that $4^{x_1} = 5$, $5^{x_2} = 6$, $6^{x_3} = 7$, ..., $127^{x_{124}} = 128$. What is $x_1 x_2 \cdots x_{124}$?

2. Two distinct numbers a and b are chosen randomly from the set $\{2, 2^2, 2^3, \ldots, 2^{20}\}$. What is the probability that $\log_a b$ is an integer?

3. To write out 31415! in full, how many digits are needed? What are the first three (most significant) digits?

4. There exist positive integers X, Y and Z, with no common factor greater than 1, such that
$$X \log_{200} 5 + Y \log_{200} 2 = Z.$$
Find the ordered triple (X, Y, Z).

5. Evaluate
$$\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdots \log_{999} 1000 \left(\log \frac{2}{3} + \log \frac{3}{4} + \log \frac{4}{5} + \cdots + \log \frac{1023}{1024} \right).$$

1.14 Focus, directrix, tangents, and parabola (part 4)

(Most of the problems in this section are selected from PEA Math 2 materials, with Richard Parris as the main author.)

1. A parabola separates the plane into two parts. The part containing the focus is the *inside part*. Let $\mathcal{P}$ be the parabola with focus $(0, 4)$ and directrix $y = -3$.
   (a) Determine, with justification, if $(4, 1.5)$ is on, inside, or outside of parabola $\mathcal{P}$.
   (b) Find an equation of $\mathcal{P}$.
   (c) Find an equation of the line tangent to $\mathcal{P}$ at $Q = (5, b)$.
2. State and explain a general process of finding the line that is tangent to a point on a parabola.
3. Verify that the point $A = \left(8, \frac{25}{3}\right)$ lies on the parabola whose focus is $(0, 6)$ and whose directrix is the x-axis. Find an equation for the line that is tangent to the parabola at A.
4. Line ℓ is tangent to parabola $y = (x-2)(x+6)$ at $(4, 20)$.
   (a) Find the focus and directrix of the parabola and then find an equation of ℓ.
   (b) Assume that $y = m(x-4) + 20$. Find m by considering the discriminant of a certain quadratic equation.
   (c) Note that $y = (x-2)(x+6) = x^2 + 4x - 12 = (x-4)^2 + 12x - 28$. Hmm ... .
5. Find an equation of a line passing through the origin that is tangent to $y = x^2 - 3x + 4$.

1.15 Sums and products (part 2)

1. Factor $abc + bcd + cde + def + efa + fab + ace + bdf$.

2. In acute triangle ABC, H is the foot of the perpendicular from A to side BC. Let $AB = c$, $BC = a$, and $CA = b$.

   (a) Express each of BH and CH in terms of a, b, c.

   (b) Express $AH \cdot BC$ as the product of four degree 1 polynomials of a, b, c.

   (c) Establish Heron's formula.

3. Find a polynomial $p(x)$ with integer coefficients with $\sqrt{2} + \sqrt[3]{3}$ as one of its roots.

4. Note that adding the following expression leads to the closed form of $1 + 2 + \cdots + n$.

$$\begin{aligned}
(0+1)^2 &= 0^2 + 2 \cdot 0 \cdot 1 + 1^2 \\
(1+1)^2 &= 1^2 + 2 \cdot 1 \cdot 1 + 1^2 \\
(2+1)^2 &= 2^2 + 2 \cdot 2 \cdot 1 + 1^2 \\
\cdots & \cdots \cdots \\
(n+1)^2 &= n^2 + 2 \cdot n \cdot 1 + 1^2
\end{aligned}$$

   One might think this is a very convoluted way. But this is a very effective way that can be generalized. Use this technique to find a compact form of the series $\sum_{k=1}^{n} k^2$.

5. Rewrite the sum

$$1 \cdot 99 + 2 \cdot 98 + 3 \cdot 97 + \cdots + 98 \cdot 2 + 99 \cdot 1.$$

   using sigma notation and then evaluate it.

1.16 Focus, directrix, tangents, and parabola (part 5)

(Most of the problems in this section are selected from PEA Math 2 materials, with Richard Parris as the main author.)

1. Consider the parabola whose focus is $F = (1, 4)$ and whose directrix is the line $x = -3$.
   (a) Sketch the parabola, and make calculations that confirm that $P = (7, 12)$ is on it.
   (b) Find the slope of the line μ through P that is tangent to the parabola.
   (c) Calculate the size of the angle that μ makes with the line $y = 12$.
   (d) Calculate the size of the angle that μ makes with segment FP.
2. Explain why it makes sense for a car headlight or a spotlight to have a bulb at the focus of a parabolic reflector.
3. Point $P = (m, n)$ lies above the line $y = x$. Point Q lies on the line $y = x$ such that PQ is parallel to the y-axis. Let R be the point on the line $y = x$ that is closest to P.
   (a) Draw an accurate diagram of the problem.
   (b) Find the coordinates of Q in terms of m and n.
   (c) Find the length of PQ in terms of m and n.
   (d) Find the measure of $\angle PQR$.
   (e) Find the length of PR (which is also the distance from P to the line $y = x$) in terms of m and n.
4. Find an equation of the curve consisting of all the points that are equidistant from point $F = (0, 4)$ and line $y = x$ and find an equation of the line tangent to this curve at $x = 1$.
5. Show that any graph of a quadratic function $y = ax^2 + bx + c$ is a parabola (if a is nonzero) by finding its focus and directrix in terms of a, b, and c.

1.17 Bouncing balls and the domain and the range of the infinite geometric series (part 3)

(Most of the problems in this section are selected from PEA Math 3 materials, with Richard Parris as the main author.)

1. The rebound ratio of a speckled green superball is 80%. It is dropped from a height of 16 feet. Consider the instant when ball strikes the ground for the fiftieth time.
   (a) How far downward has the ball traveled at this instant?
   (b) How far (upward and downward) has the ball traveled at this instant?
   (c) How much time passes between the initial drop and this instant?
2. (Continuation) How far would the ball travel if you just let it bounce and bounce and bounce . . . ? How much time does all this bouncing actually take?
3. A superball is dropped from a height of h feet, and left to bounce forever. The rebound ratio of the ball is r. In terms of r and h, find formulas for
   (a) the total distance traveled by the ball;
   (b) the total time needed for all this bouncing to take place.
4. (Continuation) Does the ball bounce *forever*? There are more than one way to interpret and answer this question.
5. What are the possible values of the sum of the geometric series $1 + t + t^2 + t^3 + \cdots$; that is, what are the other possible values to which the series can converge?

1.18 Revisit the quadratic function and its graph (part 3)

1. A function f will be called *repetitive* if there are at least two different values of x in the interval $[0, 1]$ for which $f(x)$ has the same value. What are all real numbers b for which $f(x) = x^2 + bx + 3$ is repetitive?

2. What does a parabola look like from a great distance?

3. *The reflection property of parabolas.* Let F be the focus of a parabola, and let P be an arbitrary point on the parabola. Let λ be the line through P that is parallel to the axis of symmetry of the parabola; this means that λ intersects the directrix perpendicularly at a point N. Let λ be the perpendicular bisector of FN.

   (a) Explain why P is on λ.

   (b) Explain why λ is tangent to the parabola.

   (c) Explain why λ bisects angle FPN.

   (d) Explain why it makes sense for a solar oven, a satellite dish, or a parabolic microphone to have a parabolic reflector.

4. Find an equation of a line passing through the origin that is tangent to $y = 4x^2 - 3x + 9$.

5. The distance from a point to a circle is defined as the minimum distance from the point to all the points on the circle. Find an equation describing all the points that are equidistant to circle $x^2 + y^2 - 4x + 7y = 4$ and line $y = 10$.

1.19 Fractal and recursive relation (part 2)

1. An equilateral triangle of unit area is painted step-by-step as follows: Step 1 consists of painting the triangle formed by joining the midpoints of the sides. Step 2 then consists of applying the same midpoint-triangle process to each of the three small unpainted triangles. Step 3 then consists of applying the midpoint-triangle process to each of the nine very small unpainted triangles. The result is shown at right. In general, each step consists of applying the midpoint-triangle process to each of the (many) remaining unpainted triangles left by the preceding step. Let P_n be the area that was painted *during* step n, and let U_n be the total unpainted area left after n steps have been completed.

   (a) Find U_1, U_2, U_3, P_1, P_2, and P_3.

   (b) Write a recursive description of U_n in terms of U_{n-1}. Find an explicit formula for U_n.

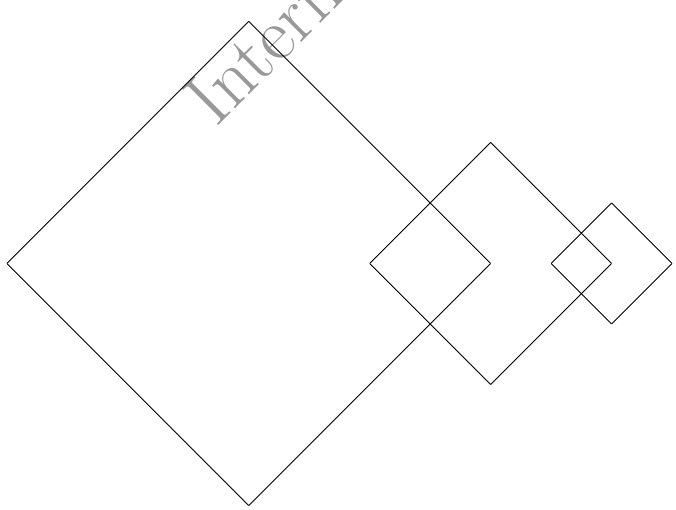
   (c) Write a recursive description of P_n in terms of P_{n-1}. Find an explicit formula for P_n.

   (d) Use your work to evaluate the sum $\frac{1}{4} + \frac{3}{16} + \frac{9}{64} + \cdots + \frac{3^{99}}{4^{100}} + \frac{3^{100}}{4^{101}}$.

   (e) Express the series of part (d) using sigma notation.

2. In the sequence $2001, 2002, 2003, \ldots,$ each term after the third is found by subtracting the previous term from the sum of the two terms that precede that term. For example, the fourth term is $2001 + 2002 - 2003 = 2000$. What is the 2004^{th} term in this sequence?

3. Square S_1 is 1×1. For $i \geq 1$, the lengths of the sides of square S_{i+1} are half the lengths of the sides of square S_i, two adjacent sides of square S_i are perpendicular bisectors of two adjacent sides of square S_{i+1}, and the other two sides of square S_{i+1} are the perpendicular bisectors of two adjacent sides of square S_{i+2}. Let $\mathcal{R}$ denote region consisting of points lying in at least one of $S_1, S_2, \ldots, S_{10}$. Find the total area of $\mathcal{R}$.



4. Ten women sit in 10 seats in a line. All of the 10 get up and then reseat themselves using all 10 seats, each sitting in the seat she was in before or a seat next to the one she occupied before. In how many ways can women be reseated?

5. Let $H_1 = A_1A_2A_3A_4A_5A_6$ be a regular hexagon with $A_1A_2 = 1$. The common region of equilateral triangles $A_1A_3A_5$ and $A_2A_4A_6$ is another regular hexagon $H_2 = A_1'A_2'A_3'A_4'A_5'A_6'$ where segment A_1A_3 intersects with segments A_2A_6 and A_2A_4 at A_1' and A_2', respectively. Triangular regions $A_1A_2A_1'$, $A_2A_3A_2'$, $\ldots$, $A_5A_6A_5'$, $A_6A_1A_6'$ are colored red. Apply the similar procedure to H_2 and so on. Find the total area of the colored region.

1.20 Revisit the quadratic function and its graph (part 4)

1. The parabola $y = ax^2 + bx + c$ has vertex (p, p) and y-intercept $(0, -p)$, where $p \neq 0$. What is b?

2. Let x_1 and x_2 be the two zeros of $f(x) = x^2 + bx + c$. If $|x_1 - x_2| = 3$, find the minimum of $f(x)$.

3. What are all possible values of the real numbers a for which the roots of $x^2-(a+1)x+a+4 = 0$ are

   (a) Both negative?

   (b) Both positive?

   (c) Both greater than 2?

   (d) One negative, one positive?

   (e) One less than 1, one greater than 1?

4. Justify the statement: *"All parabolas are similar"*.

5. If $x^2 + y^2 - 30x - 40y + 576 = 0$, find the largest possible value of $\dfrac{y}{x}$.

Chapter 2

Algebra Challenges

2.1 Quadratic function (part 1)

1. If a and b are the solutions to the equation $x^2 - 5x + 9 = 0$, what is the value of $(a-1)(b-1)$? Express $(a-c)(b-c)$ in terms of c.

2. Find all real values of x for which

$$\frac{1}{\sqrt{x}+\sqrt{x-2}} + \frac{1}{\sqrt{x+2}+\sqrt{x}} = \frac{1}{4}.$$

3. A friend of Taylor, who studies calculus, told her that the area of the region R_1 bounded by the x-axis and the graph of $y = \frac{9}{2} - x^2$ is equal to $9\sqrt{2}$. Taylor thought that she could find a good approximation of the area of this region if she computes the area of the region R_2 bounded by the x-axis and the graph of $y = 4 - \lfloor x^2 \rfloor$. Explain the reasoning behind Taylor's idea and find the area of R_2. Find the percent error of the approximation.

4. Let a and b be the two real solutions to $8x^2 + 9x + c = 0$, where c is a given positive constant. For some real number r, the value of the expression

$$\frac{(a-\sqrt{c})(b-\sqrt{c})}{c+c^r}$$

does not depend on c. Find r and find this value.

5. Recall that the Fibonacci numbers are defined as $f_1 = 1$, $f_2 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for all $n \geq 2$. Express $3f_{2012}^2 - 4f_{2012}f_{2010} + f_{2010}^2$ as a product of two Fibonacci numbers; i.e., write your answer in the form $f_m f_n$.

2.2 Practice set 1

1. After setting Prep record for the 60-meter hurdles, Ogechi sets a PEA career goal: to run this race 0.75 seconds faster. Ogechi calculated that this means a rate increase of a third meters per second. Figure out Ogechi's Prep record time.

2. Let A, R, M, and L be positive real numbers such that

$$\begin{aligned} \log(A \cdot L) + \log(A \cdot M) &= 2 \\ \log(M \cdot L) + \log(M \cdot R) &= 3 \\ \log(R \cdot A) + \log(R \cdot L) &= 4. \end{aligned}$$

   Compute the value of the product $A \cdot R \cdot M \cdot L$.

3. Find the sum of the squares of the solutions to

$$\left| x^2 - x + \frac{1}{2008} \right| = \frac{1}{2008}.$$

4. Determine all triples (a, b, c) of distinct real numbers such that each of the three equations $ax^2 + 2bx + c = 0$, $bx^2 + 2cx + a = 0$, and $cx^2 + 2ax + b = 0$ has exactly one real solution.

5. The number

$$A = \sqrt{31 + 4\sqrt{15} + 8\sqrt{10} + 4\sqrt{6}}$$

   can be written as $a\sqrt{u} + b\sqrt{v} + c\sqrt{w}$, where a, b, c, u, v, and w are positive integers. Find $a + b + c + u + v + w$.

2.3 Challenges on distance and motion

1. Pat walks to Kim's house on a route that is 3 miles long. Pat jogs home on a route that is 5 miles long, at a speed that is 4 miles per hour faster than it is when walking. The total time for the roundtrip is an hour and 45 minutes. Find Pat's walking speed in miles per hour.

2. A 100 foot long moving walkway moves at a constant rate of 6 feet per second. Al steps onto the start of the walkway and stands. Bob steps onto the start of the walkway two seconds later and strolls forward along the walkway at a constant rate of 4 feet per second. Two seconds after that, Cy reaches the start of the walkway and walks briskly forward beside the walkway at a constant rate of 8 feet per second. At a certain time, one of these three is exactly halfway between the other two. At that time, find the distance in feet from the start of the walkway of the middle person.

3. Ed and Sue bike at equal and constant rates. Similarly, they jog at equal and constant rates, and they swim at equal and constant rates. Ed covers 74 kilometers after biking for 2 hours, jogging for 3 hours, and swimming for 4 hours, while Sue covered 91 kilometers after jogging for 2 hours, swimming for 3 hours, and biking for 4 hours. Their biking, jogging, and swimming rates are all whole numbers of kilometers per hour. Find the sum of the squares of Ed's biking, jogging, and swimming rates.

4. Four steps by Dog *Kat* can exactly cover the distance of 7 steps by Cat *Fat*. The time for Kat running 6 steps is equal to that of Fat running 5 steps. Supposing that Fat is 55 yards away before Kat starts to catch Fat, how far can Fat run before it gets caught?

5. Every day, Chris takes the train home from school. His train gets into the station at 3:00 P.M. Corey drives out to pick Chris up, arriving at the station just as Chris does at 3:00 P.M.

   One particular day, Chris has early release at school and takes an earlier train, arriving at the station instead at 2:00 P.M. Lacking a cell phone, he starts walking in the direction to home. Chris meets Corey along the way on Corey's way to the station. They drive to home immediately, getting there 20 minutes earlier than usual.

   Assuming constant rates of walking and driving, how long was Chris walking? (In order to make this problem more entertaining, the conditions are not mathematically strictly defined. In order to solve the problem, one has to make certain assumptions with common sense. What the assumptions you make?)

2.4 Practice set 2

1. Define a sequence of real numbers $a_1, a_2, a_3, \ldots$ by $a_1 = 1$ and $a_{k+1}^3 = 81a_k^3$ for all $k \geq 1$. Then $a_{100} = m^n$ for some positive integers m and n. What is the minimum value of $m + n$?

2. A cylinder with radius r and height h has volume 1 and total surface area 12. Compute $\frac{1}{r} + \frac{1}{h}$.

3. Find $(\log_2 x)^2$ if $\log_2(\log_8 x) = \log_8(\log_2 x)$.

4. The number
$$B = \sqrt{31 + 4\sqrt{15} + 6\sqrt{10} + 10\sqrt{6}}$$
can be written as $a\sqrt{u} + b\sqrt{v} + c\sqrt{w}$, where a, b, c, u, v, and w are positive integers. Find $a + b + c + u + v + w$.

5. Given that a, b, c are real numbers that satisfy $a^2 + 2b = 7$, $b^2 + 4c = -7$, and $c^2 + 6a = -14$, compute $a^2 + b^2 + c^2$.

2.5 Rational and radical expressions (part 1)

1. Evaluate
$$\frac{\left(\frac{1}{a}+\frac{1}{b}\right)^3}{\left(\frac{1}{a}-\frac{1}{b}\right)^2},$$
where real numbers a and b satisfy $4a^2 - 4a + \sqrt{1-ab} + 1 = 0$.

2. Find a fourth degree polynomial $p(x)$ with integer coefficients such that $\sqrt{2}+\sqrt{3}+\sqrt{6}$ is one of its roots.

3. Evaluate $\sqrt[3]{20+14\sqrt{2}} + \sqrt[3]{20-14\sqrt{2}}$.

4. If $a-b = 2+\sqrt{3}$ and $b-c = 2-\sqrt{3}$, find the value of $a^2+b^2+c^2-ab-bc-ca$.

5. There is a constant c and an interval $[a,b]$ for which
$$\sqrt{x+\sqrt{2x-1}} + \sqrt{x-\sqrt{2x-1}} = c$$
whenever $a \le x \le b$. Find the constant c, and find a and b so that $b-a$ is maximum.

2.6 Practice set 3

1. How many real numbers x are solutions to the following equation?

$$|x-1| = |x-2| + |x-3|.$$

2. Positive integers are arranged in a table as follows:

1	3	6	10	...
2	5	9	...	
4	8	...		
7	...			
...				

In the table, rows are numbered from the top to bottom, and the columns are numbers from left to right. (For example, 6 is in row 1 column 3, 8 is in row 3 column 2.) Given that 1000 is in row m and column n, find the ordered pair (m, n).

3. If x and y are positive real number for which

$$(x+y)^2 + (x-y)^2 = 10 \quad \text{and} \quad (x+y)^4 + (x-y)^4 = 98,$$

what is xy?

4. Determine if $\sqrt[3]{9-\sqrt{80}} + \sqrt[3]{9+\sqrt{80}}$ is a rational number.

5. The graph of $y = a|x+b|+c$ passes through points $A = (12, -1)$, $B = (7, -3)$, and $C = (5, 1)$. The graphs of $y = a|x+b|+c$ and $y = -2|x-9|+12$ enclose a parallelogram. Find the area of the parallelogram.

2.7 Revisit Vieta's relation for the quadratic equation

1. If a and b are two distinct real numbers that satisfy $a^2 = 4a + 3$ and $b^2 = 4b + 3$, find the value of $\frac{a^2}{b} + \frac{b^2}{a}$.

2. The quadratic equation $x^2 + mx + n = 0$ has roots that are twice those of $x^2 + px + m = 0$, and none of m, n, and p is zero. What is the value of n/p?

3. Consider the equation $x^2 - 3x - 5 = 0$. Write a equation whose solutions are

   (a) the opposites of the solutions of the given equation;

   (b) the reciprocals of the solutions of the given equation;

   (c) n times the solutions of the given equation;

   (d) one more than the solutions of the given equation;

   (e) the cubes of the solutions of the given equation.

4. Line ℓ has slope 1 and passes through point $(1, 3)$. Line ℓ intersects the circle $x^2 + y^2 = 6$ at points A and B. Find

   (a) the midpoint of segment AB;

   (b) the length of segment AB.

5. Let m and n, $m > n$, be the solutions to $11x^2 = 21x + 11$. Find the value of

$$\frac{n(1 - m^3)}{1 - m} + 1.$$

2.8 Practice set 4

1. Find all integers n for which $\dfrac{n^3+8}{n^2-4}$ is an integer.

2. Let a and b be positive real numbers such that $\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{5}{a+b}=0$. Find the value of $\dfrac{a^3}{b^3}+\dfrac{b^3}{a^3}$.

3. Determine, with justification, the number of terms in the sequence $(\frac{8}{9})^1, (\frac{8}{9})^2, (\frac{8}{9})^3, \ldots$ each has its value starting with exactly 2001 zeros after the decimal place.

4. The graphs of $y=\log_3 x$, $y=\log_x 3$, $y=\log_{\frac{1}{3}} x$, and $y=\log_x \frac{1}{3}$ are plotted on the same set of axes. How many points in the plane with positive $x-$coordinates lie on two or more of the graphs?

5. Find the area of the portion of the xy-plane enclosed by the curve with equation

$$\left|x-\frac{1}{2}\right|+\left|x+\frac{1}{2}\right|+\frac{2|y|}{\sqrt{3}}=2.$$

2.9 A short review on sums and products (part 1)

1. Expand $(a+b+c)(ab+bc+ca)-(a+b)(b+c)(c+a)$ and $(a+b+c)^3$.

2. Compute
$$\left\lfloor \frac{2007!+2004!}{2006!+2005!} \right\rfloor.$$
(Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

3. The years from 1900 to 1999 are written down consecutively and then plus and minus signs are placed alternately between the digits as shown:
$$1+9-0+0-1+9-0+1-1+9-0+2-\cdots-1+9-9+9=k.$$
Compute the value of k.

4. Suppose that x, y, and z are three positive numbers that satisfy the equations
$$xyz=1, \quad x+\frac{1}{z}=5, \quad y+\frac{1}{x}=29.$$
Compute $z+\dfrac{1}{y}$.

5. Let a, b, and c be real numbers such that $a-7b+8c=4$ and $8a+4b-c=7$. Show that the $a^2-b^2+c^2$ value is constant and find this constant.

2.10 Practice set 5

1. The roots of $ax^2+bx+c=0$ are irrational, but their calculator approximations are 0.8430703308 and -0.5930703308. If a, b, and c are integers whose greatest common divisor is 1 and which satisfy $a > 0$, $|b| \le 10$ and $|c| \le 10$, compute the ordered triple (a, b, c).

2. Find all pairs of nonintersecting sets (A, B) such that the union of A and B is the set $\{1, 2, \ldots, 10\}$ and the product of the elements in A is equal to the sum of the elements in B.

3. For some positive real number a, the graph of $y = ax^2 + bx + c$ passes through points $(-1, 0)$ and $(0, 1)$.

   (a) Determine the signs of b and c;

   (b) Find the range of $a + b + c$.

4. The expression $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Find the value of

$$\left\lfloor \frac{2002!}{2001! + 2000! + 1999! + \cdots + 1!} \right\rfloor.$$

5. Positive integers a and b satisfy the relation

$$\log_a b = (\log_2 3)(\log_6 7) + \log_2 3 + \log_6 7.$$

   Find the minimum value of ab.

2.11 Rational and radical expressions (part 2)

1. Expand

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(\sqrt{a}+\sqrt{b}-\sqrt{c}\right)\left(\sqrt{b}+\sqrt{c}-\sqrt{a}\right)\left(\sqrt{c}+\sqrt{a}-\sqrt{b}\right).$$

How do you know the result you obtain is consistent with some of our earlier result? Then rationalize the denominator of

$$\frac{1}{\sqrt{3}+\sqrt{5}+\sqrt{7}}$$

2. Given that $x = 2-\sqrt{3}$, compute

$$\frac{x^4-4x^3-x^2+9x-4}{x^2-4x+5}.$$

3. Solve the following equations for x.

   (a) $(\log x)^2+\log(x^3)+2=0$

   (b) $\sqrt{x^2-5x+1}+\sqrt{x^2-6x+6}=2$

4. Let a, b, c be nonzero real numbers with

$$\frac{a+b-c}{c}=\frac{b+c-a}{a}=\frac{c+a-b}{b}.$$

Determine all possible values of $(a+b)(b+c)(c+a)/abc$.

5. Find a polynomial $p(x)$ with integer coefficients with $\sqrt[3]{4}+\sqrt[3]{2}+1$ as one of its roots.

2.12 Practice set 6

1. If a and b are positive real numbers and each of the equations $x^2+ax+2b=0$ and $x^2+2bx+a=0$ has real roots. What is the minimum value of $a+b$?

2. Consider the sequence $a_n = \dfrac{2n-7}{2^n}$ for $n = 1, 2, \ldots$. Let M and m be the maximum and minimum values of the terms in the sequence. Find $M+m$.

3. Find the smallest positive integer k such that $1^2 + 2^2 + \cdots + k^2$ is a multiple of 200.

4. If $a^2 - a + 1 = 0$, find the value of $a^{2009} + \dfrac{1}{a^{2009}}$.

5. Find all possible values of a, where $a > 0$ and $a \neq 1$, such that $f(x) = \log_a |ax^2 - x|$ is increasing on the interval $[3, 4]$.

2.13 Quadratic equation with integer roots

1. Let p and q be primes such that $x^2 - px + 2q$ has two integer roots. Find the greatest value of $p < 100$ for which such pair exists.

2. The polynomial $x^3 - ax^2 + bx - 2010$ has three positive integer zeros. What is the smallest possible value of a?

3. Determine all possible values of integer a such that the roots of

$$(a+1)x^2 - (a^2+1)x + 2a^3 - 6 = 0$$

are integers.

4. Quadratic equation $x^2 + ax + b + 1 = 0$ has two non-zero integer roots. Prove that $a^2 + b^2$ can never be a prime.

5. Let n be a given positive integer, and let r_1 and r_2 denote the roots of the equation $x^2 + 2(n+1)x + 6n - 5 = 0$. Find the equation with roots $\lfloor r_1 \rfloor$ and $\lfloor r_2 \rfloor$. (For real number x, $\lfloor x \rfloor$ denote the greatest integer less than or equal to x.)

2.14 Practice set 7

1. Find all three different positive integers x such that $4^x + 4^{1993} + 4^{1996}$ is a perfect square.
2. Find the least integer greater than $(7 + \sqrt{40})^3$.
3. What is the sum of the reciprocals of the roots of the equation
$$\frac{2003x}{2004} + 1 + \frac{1}{x} = 0?$$
4. If $f(x) = (x-3)^2 - 1$, compute the set of real numbers such that $f(|x|) = |f(x)|$.
5. Let a and b be positive integers satisfying $\dfrac{ab+1}{a+b} < \dfrac{3}{2}$. The maximum possible value of $\dfrac{a^3b^3+1}{a^3+b^3}$ is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

2.15 A short review on sums and products (part 2)

1. Positive integers a, b, c, and d satisfy $a > b > c > d$, $a+b+c+d = 2010$, and $a^2 - b^2 + c^2 - d^2 = 2010$. Find the number of possible values of a.

2. We have derived a compact form of the series $\sum_{k=1}^{n} k^2$ in two different approaches. Now derive a compact form of the series $\sum_{k=1}^{n} k^3$ in two different approaches.

3. For any positive integer x, denote by $S(x)$ the sum of the digits of x, and let
$$T(x) = |S(x+2) - S(x)|.$$
For example, $T(199) = |S(201) - S(199)| = |3 - 19| = 16$. How many values $T(x)$ do not exceed 1999?

4. Rewrite the sum
$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \cdots + 99 \cdot 100 \cdot 101$$
using sigma notation and then evaluate it.

5. Let $S = \{2^0, 2^1, 2^2, \ldots, 2^{10}\}$. Consider all possible positive differences of pairs of elements of S. Let N be the sum of all of these differences. Find the remainder when N is divided by 1000.

2.16 Practice set 8

1. Rounded to the nearest hundredth, the positive real number x satisfying the equation $3^x + 6x = 99$ is given by $x = 3.93$. Find the solution to the equation $3^x + 2x = 31$, rounding your answer to the nearest hundredth.

2. The x-intercepts of the equation $y = x^2 + px + q$ are positive integers. Given that $p + q = 100$, find all lines that can be axis of symmetry of the parabola.

3. Construction workers want to place a circular pipe on the bottom of a trench in a shape of a parabola which is 8 meters wide and 8 meters deep. Show that workers will not be able to place a pipe having 4 meters in diameter. Find the largest diameter of the pipe that is possible to place on the bottom of the trench.

4. Alice, Bob, and Charlie are visiting Princeton and decide to go to the Princeton U-Store to buy some tiger plushies. They each buy at least one plushie at price p. A day later, the U-Store decides to give a discount on plushies and sell them at p' with $0 < p' < p$. Alice, Bob, and Charlie go back to the U-Store and buy some more plushies with each buying at least one again. At the end of that day, Alice has 12 plushies, Bob has 40, and Charlie has 52 but they all spent the same amount of money: \$42. How many plushies did Alice buy on the first day?

5. Find the unique positive integer n such that

$$2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + 5 \cdot 2^5 + \cdots + n \cdot 2^n = 2^{n+10}.$$

2.17 Quadratic function (part 2)

1. Alex was asked to solve the equation $x^2+bx+c=0$, where b and c are integers. Being absent minded, Alex inadvertently solved the equation $x^2+cx+b=0$. One of the roots obtained is the same as a root of the original equation, but the second root is m less than the second root of the original equation. Express b and c in terms of m.

2. Find the extreme values (the maximum and the minimum) of $4x^2+3xy+y^2-6x-3y$ if $x \ge 0$, $y \ge 0$, and $2x+y=6$.

3. Given that $x^2+(a-8)x+12-ab=0$ has real solutions for all real numbers a, find the range of real number b.

4. Determine the maximum value of a for which there is at least one real solution (x, y) to the system
$$x^2+y^2=1 \quad \text{and} \quad x^2y^2=a.$$
Once you determine the maximum value of a, draw a graph that supports your answer in the extremal case.

5. Solve
$$\frac{1}{x^2+11x-8}+\frac{1}{x^2+2x-8}+\frac{1}{x^2-13x-8}=0.$$

2.18 Practice set 9

1. Find the sum of all real numbers x such that $\log_2\left(56 - 2^{3+x}\right) = 4 - x$.

2. The combined weight of two airline passengers' checked luggage is 105 pounds. The airline's checked-luggage policy allows each passenger a maximum weight at no charge, and charges a constant, per pound weight for any extra weight. One passenger pay \$12 for the total weight of her luggage, and the other passenger pays \$18 for the total weight of his luggage. If all of the luggage had belonged to one of the passenger, the over-weight charge would have been \$78. How many pounds of luggage is one person permitted without charge by the airline?

3. Let $g(x, y) = x^2 - 4xy + 5y^2 + 10y + 28$. Find the minimum value of $g(x, y)$, and the values of x and y at which this minimum is achieved.

4. The graph of $y^2 + 2xy + 40|x| = 400$ partitions the plane into several regions. What is the area of the bounded region?

5. Consider the sequence $a_0 = a_1 = 1$, and $a_{n+2} = 2a_{n+1} - a_n + 2$ for $n \geq 0$. Verify that $a_{n^2+1} = a_{n+1}a_n$ for $n = 0, 1, 2, 3$. Establish how fast does the sequence grow: is it linear, quadratic, exponential? Find the closed form expression for the general term of the sequence, a_n, and show that $a_{n^2+1} = a_{n+1}a_n$ for all positive integers n.

2.19 A short review on sums and products (part 3)

1. Given that
$$x = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}} \quad \text{and} \quad y = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}},$$
evaluate
$$\frac{2\sqrt{10(x+y)} - 3\sqrt{xy}}{3x^2 - 5xy + 3y^2}.$$

2. Let a, b, x, y be real number with $ax+by = 2007$ and $ay-bx = 2009$, compute $(a^2+b^2)(x^2+y^2)$.

3. Find the general formula for the sum
$$1 + 11 + 111 + \ldots + \underbrace{1\ldots1}_{n \text{ digits}}.$$

4. Express $\displaystyle\sum_{k=1}^{n}(-1)^{k+1}k^2$ in closed form.

5. A sequence of positive integers with $a_1 = 1$ and $a_9 + a_{10} = 646$ is formed so that the first three terms are in geometric progression, the second, third, and fourth terms are in arithmetic progression, and, in general, for all $n \ge 1$, the terms a_{2n-1}, a_{2n}, and a_{2n+1} are in geometric progression, and the terms a_{2n}, a_{2n+1}, and a_{2n+2} are in arithmetic progression. Let a_n be greatest term in this sequence that is less than 1000. Find $n + a_n$.

2.20 Practice set 10

1. Find all real solutions (x, y) to the system of equations:

$$\begin{aligned} x^3 - y^3 &= 728, \\ x^2y - xy^2 &= 240. \end{aligned}$$

2. Given that a and b are positive real numbers satisfying the equation

$$2 + \log_2 a = 3 + \log_3 b = \log_6(a + b),$$

compute $1/a + 1/b$.

3. In order to complete a large job, 1000 workers were hired, just enough to complete the job on schedule. All the workers stayed on the job while the first quarter of the work was done, so the first quarter of the work was completed on schedule. Then 100 workers were laid off, so the second quarter of the work was completed behind schedule. Then an additional 100 workers were laid off, so the third quarter of the work was completed still further behind schedule. Given that all workers work at the same rate, what is the minimum number of additional workers, beyond the 800 workers still on the job at the end of the third quarter, that must be hired after three-quarters of the work has been completed so that the entire can be completed on schedule or before?

4. Compute $\lfloor 100000(1.002)^{10} \rfloor$.

5. Consider the graph $\mathcal{C}$ of $y = x^2 + (a + 1)x + \dfrac{2a + 1}{4}$, where a is a real number.

   (a) Show that $\mathcal{C}$ passes through a fixed point (independent to the choice of a).

   (b) Determine the locus of the vertex of $\mathcal{C}$ as a ranges over the real numbers.