1.7 Practice with binomial coefficients (part 1)

1. The PEA mathematics department is to hold a meeting to discuss pedagogy. After a long conversation among 23 members of the department, they decide to split into 5 groups of three and 2 groups of four to continue their discussion. In how many ways can this be done?

2. In the expansion of $(ax+b)^{2000}$, where a and b are relatively prime positive integers, the coefficients of x^2 and x^3 are equal. Find $a+b$.

3. How many ways are there to place two `A`'s, two `B`'s, two `C`'s, and two `D`'s in four distinguishable boxes such that every box has two letters?

4. What is the value of the constant term in the expansion of $\left(\left(x+\frac{1}{x}\right)^2-4\right)^{20}$?

5. There are 10 people who want to choose a committee of 5 people among them. They do this by first electing a set of 1, 2, 3, or 4 committee leaders, who then choose among the remaining people to complete the 5-person committee. In how many ways can the committee be formed, assuming that people are distinguishable? (Two committees that have the same members but different sets of leaders are considered to be distinct.)

6. A coin is altered so that the probability that it lands on heads is less than $\frac{1}{2}$ and when the coin is flipped four times, the probability of an equal number of heads and tails is $\frac{1}{6}$. What is the probability that the coin lands on heads?

7. Let $A=\{a_1,a_2,\dots,a_{100}\}$ and $B=\{1,2,\dots,50\}$. Determine the number of surjective functions f from A to B such that $f(a_1)\le f(a_2)\le\cdots\le f(a_{100})$. What if f does not need to be surjective?

8. Given that
$$\frac{1}{2!17!}+\frac{1}{3!16!}+\frac{1}{4!15!}+\frac{1}{5!14!}+\frac{1}{6!13!}+\frac{1}{7!12!}+\frac{1}{8!11!}+\frac{1}{9!10!}=\frac{N}{1!18!}.$$
find the greatest integer that is less than $N/100$.

9. The expression $(x+y+z)^{2006}+(x-y-z)^{2006}$ is simplified by expanding it and combining like terms. How many terms are in the simplified expression?

10. For $k\ge 3$, we define an ordered k-tuple of real numbers $(x_1,x_2,\dots,x_k)$ to be *special* if, for every i such that $1\le i\le k$, the product $x_1x_2\cdots x_k=x_i^2$. Compute the smallest value of k such that there are at least 2009 distinct special k-tuples.

1.10 Let's count (part 4)

1. How many subsets A of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ have the property that no two elements of A sum to 11?

2. When a four-digit number with units digit 2 is divided by a one-digit number, it has a remainder of 1. How many such four-digit numbers are there?

3. Say that an ordered triple (a, b, c) is *pleasing* if

   (a) a, b, and c are in the set $\{1, 2, \ldots, 17\}$;

   (b) both $b - a$ and $c - b$ are greater than 3;

   (c) at least one the numbers $b - a$ and $c - b$ is equal to 4.

   How many pleasing triples are there? How many triples are there if we remove condition (c)?

4. Twelve chairs are set up in a row for the Princeton garlic-eating contest. Only five eaters attend the competition, but none will sit next to any other. In how many ways can the eaters be seated? What if there are twenty chairs and eight eaters?

5. An integer is called *snakelike* if its decimal representation $a_1a_2a_3 \ldots a_k$ satisfies $a_i < a_{i+1}$ if i is odd and $a_i > a_{i+1}$ if i is even. How many snakelike integers between 1000 and 9999 have four distinct digits?

6. Twelve fair dice are rolled. What is the probability that the product of the numbers on the top faces is prime?

7. Boston Yankees and New York Red Sox play a series. The first team to win three games wins the series. Each team is equally likely to win each game, there are no ties, and the outcomes of the individual games are independent. If Boston Yankees wins the second game and New York Red Sox wins the series, what is the probability that Boston Yankees wins the first game?

8. How many positive integers less than or equal to 240 can be expressed as a sum of distinct factorials? Consider 0! and 1! to be distinct.

9. When rolling a certain unfair six-sided die with faces numbered 1, 2, 3, 4, 5 and 6, the probability of obtaining face F is greater than 1/6, the probability of obtaining the face opposite F is less than 1/6, the probability of obtaining each of the other face is 1/6, and the sum of the numbers on each pair of opposite faces is 7. When two such dice are rolled, the probability of obtaining a sum of 7 is 47/288. Given that the probability of obtain face F is m/n, where m and n are relatively prime positive integers, find $m + n$.

10. A frog is placed at the origin on the number line, and moves according to the following rule: in a given move, the frog advances to either the closest point with a greater integer coordinate that is a multiple of 3, or to the closest point with a greater integer coordinate that is a

multiple of 13. A *move sequence* is a sequence of coordinates which corresponds to valid moves, beginning with 0, and ending with 39. For example, 0, 3, 6, 13, 15, 26, 39 is a move sequence. How many move sequences are possible for the frog?