1.3 Bouncing balls and the domain and the range of the infinite geometric series (part 1)

1. A speckled green superball has a 75% rebound ratio. When you drop it from a height of 16 feet, it bounces and bounces and bounces ....

   (a) How high does the ball bounce after it strikes the ground for the third time?

   (b) How high does the ball bounce after it strikes the ground for the seventeenth time?

   (c) When it strikes the ground for the second time, the ball has traveled a total of 28 feet in a *downward* direction. Verify this. How far downward has the ball traveled when it strikes the ground for the seventeenth time?

2. (Continuation) At the top of its second rebound, the ball has traveled 21 feet upward.

   (a) At the top of its seventeenth rebound, how far upward has the ball traveled?

   (b) At the top of its seventeenth rebound, how far has the ball traveled in total?

3. (Continuation) How far would the ball travel if you just let it bounce and bounce and bounce ....

4. When $t = 4/5$, the infinite series $1 + t + t^2 + t^3 + \cdots$ *equals* or *converges* to 5. When $t = -2/3$, the series converges to 3/5. What do these statements mean? (Consider the partial sums $s_0 = 1$, $s_1 = 1 + t$, $s_2 = 1 + t + t^2$, ..., $s_n = 1 + t + \cdots + t^n$.)

5. (Continuation) For what t-values is it correct to say that the series $1 + t + t^2 + t^3 + \cdots$ has a *sum*? What is this sum?

1.14 Focus, directrix, tangents, and parabola (part 4)

1. A parabola separates the plane into two parts. The part containing the focus is the *inside part*. Let $\mathcal{P}$ be the parabola with focus $(0, 4)$ and directrix $y = -3$.
   (a) Determine, with justification, if $(4, 1.5)$ is on, inside, or outside of parabola $\mathcal{P}$.
   (b) Find an equation of $\mathcal{P}$.
   (c) Find an equation of the line tangent to $\mathcal{P}$ at $Q = (5, b)$.
2. State and explain a general process of finding the line that is tangent to a point on a parabola.
3. Verify that the point $A = \left(8, \frac{25}{3}\right)$ lies on the parabola whose focus is $(0, 6)$ and whose directrix is the x-axis. Find an equation for the line that is tangent to the parabola at A.
4. Line ℓ is tangent to parabola $y = (x - 2)(x + 6)$ at $(4, 20)$.
   (a) Find the focus and directrix of the parabola and then find an equation of ℓ.
   (b) Assume that $y = m(x - 4) + 20$. Find m by considering the discriminant of a certain quadratic equation.
   (c) Note that $y = (x - 2)(x + 6) = x^2 + 4x - 12 = (x - 4)^2 + 12x - 28$. Hmm ... .
5. Find an equation of a line passing through the origin that is tangent to $y = x^2 - 3x + 4$.

1.15 Sums and products (part 2)

1. Factor $abc + bcd + cde + def + efa + fab + ace + bdf$.

2. In acute triangle ABC, H is the foot of the perpendicular from A to side BC. Let $AB = c$, $BC = a$, and $CA = b$.

   (a) Express each of BH and CH in terms of a, b, c.

   (b) Express $AH \cdot BC$ as the product of four degree 1 polynomials of a, b, c.

   (c) Establish Heron's formula.

3. Find a polynomial $p(x)$ with integer coefficients with $\sqrt{2} + \sqrt[3]{3}$ as one of its roots.

4. Note that adding the following expression leads to the closed form of $1 + 2 + \cdots + n$.

$$\begin{aligned}(0+1)^2 &= 0^2 + 2 \cdot 0 \cdot 1 + 1^2\\ (1+1)^2 &= 1^2 + 2 \cdot 1 \cdot 1 + 1^2\\ (2+1)^2 &= 2^2 + 2 \cdot 2 \cdot 1 + 1^2\\ \cdots &\quad \cdots \quad \cdots\\ (n+1)^2 &= n^2 + 2 \cdot n \cdot 1 + 1^2\end{aligned}$$

   One might think this is a very convoluted way. But this is a very effective way that can be generalized. Use this technique to find a compact form of the series $\sum_{k=1}^{n} k^2$.

5. Rewrite the sum

$$1 \cdot 99 + 2 \cdot 98 + 3 \cdot 97 + \cdots + 98 \cdot 2 + 99 \cdot 1.$$

   using sigma notation and then evaluate it.

2.8 Algebra practice set 4

1. Find all integers n for which $\dfrac{n^3+8}{n^2-4}$ is an integer.

2. Let a and b be positive real numbers such that $\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{5}{a+b}=0$. Find the value of $\dfrac{a^3}{b^3}+\dfrac{b^3}{a^3}$.

3. Determine, with justification, the number of terms in the sequence $(\frac{8}{9})^1, (\frac{8}{9})^2, (\frac{8}{9})^3, \ldots$ each has its value starting with exactly 2001 zeros after the decimal place.

4. The graphs of $y=\log_3 x$, $y=\log_x 3$, $y=\log_{\frac{1}{3}} x$, and $y=\log_x \frac{1}{3}$ are plotted on the same set of axes. How many points in the plane with positive $x-$coordinates lie on two or more of the graphs?

5. Find the area of the portion of the xy-plane enclosed by the curve with equation

$$\left|x-\frac{1}{2}\right|+\left|x+\frac{1}{2}\right|+\frac{2|y|}{\sqrt{3}}=2.$$