Lectures on Challenging Mathematics

UC2 Practice Tests
An Invitation to Computational Mathematics

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"Cogito ergo Sum" – "I think, therefore I am"

René Descartes (1596-1650)

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Chapter 1

Practice Tests

1.1 Practice tests (part 1)

1.1.1 Selected entry level algebra problems from AIME (part 1)

1. Find the value of $a_2 + a_4 + a_6 + \cdots + a_{98}$ if a_1, a_2, a_3,... is an arithmetic progression with common difference 1, and $a_1 + a_2 + \cdots + a_{98} = 137$.

2. Let $x_1 = 1$ and let $x_n = \frac{n}{x_{n-1}}$ for $n > 1$. Find a way to compute x_{15} without computing the exact values of $x_2, x_3, \ldots, x_{14}$.

3. In a *magic square*, the sum of the three entries in any row, column, or diagonal is the same value. The figure shows four of the entries of a magic square. Find x.

x	19	96
1		

4. Find the value of
$$(52 + 6\sqrt{43})^{\frac{3}{2}} - (52 - 6\sqrt{43})^{\frac{3}{2}}.$$

5. For each real number x, let $\lfloor x \rfloor$ denote the greatest integer that does not exceed x. For how many positive integers n is it true that $n < 1000$ and that $\lfloor \log_2 n \rfloor$ is a positive even integer?

1.1.2 Selected medium to challenging level problems from AMC 2013 (part 1)

6. Jo and Blair take turns counting from 1 to one more than the last number said by the other person. Jo starts by saying "1, so Blair follows by saying "1, 2. Jo then says "1, 2, 3, and so on. What is the 53^{rd} number said?

7. Two sides of a triangle have lengths 10 and 15. The length of the altitude to the third side is the average of the lengths of the altitudes to the two given sides. How long is the third side?

8. Daphne is visited periodically by her three best friends: Alice, Beatrix, and Claire. Alice visits every third day, Beatrix visits every fourth day, and Claire visits every fifth day. All three friends visited Daphne yesterday. How many days of the next 365-day period will exactly two friends visit her?

9. A group of 12 pirates agree to divide a treasure chest of gold coins among themselves as follows. The k^{th} pirate to take a share takes $\frac{k}{12}$ of the coins that remain in the chest. The number of coins initially in the chest is the smallest number for which this arrangement will allow each pirate to receive a positive whole number of coins. How many coins does the 12^{th} pirate receive?

10. The number 2013 is expressed in the form

$$2013 = \frac{a_1!a_2!\cdots a_m!}{b_1!b_2!\cdots b_n!}$$

where $a_1 \geq a_2 \geq \cdots \geq a_m$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ are positive integers and $a_1 + b_1$ is as small as possible. What is $|a_1 - b_1|$?

1.2 Practice tests (part 2)

1.2.1 Selected entry level algebra problems from AIME (part 2)

1. Find an ordered pair (a, b) of real numbers for which $x^2 + ax + b$ has a non-real root whose cube is 343.

2. What is the respective product of the real roots of each of the following equation?

$$\sqrt[4]{x} = \frac{12}{7 - \sqrt[4]{x}} \quad \text{and} \quad x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}$$

3. The number

$$\frac{2}{\log_4 2000^6} + \frac{3}{\log_5 2000^6}$$

can be written as m/n where m and n are relatively prime positive integers. Find $m + n$.

4. Let N be the sum

$$2\left(\lceil \log_{\sqrt{2}} 2 \rceil - \lfloor \log_{\sqrt{2}} 2 \rfloor\right) + 3\left(\lceil \log_{\sqrt{2}} 3 \rceil - \lfloor \log_{\sqrt{2}} 3 \rfloor\right) + \cdots + 1000\left(\lceil \log_{\sqrt{2}} 1000 \rceil - \lfloor \log_{\sqrt{2}} 1000 \rfloor\right).$$

Find the remainder when N is divided by 1000. (Here $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x and $\lceil x \rceil$ denotes the least integer that is greater than or equal to x).

5. Let $f(x) = |x - p| + |x - 15| + |x - p - 15|$, where $0 < p < 15$. Determine the minimum value taken by $f(x)$ for x in the interval $p \le x \le 15$.

1.2.2 Selected entry level problems from Mandelbrot 2013 (part 1)

6. Suppose that Amal runs 20% faster than Shalin. If Shalin can run one lap in 84 secons, then how many *minutes* will it take Amal to run three laps?

7. Clair writes down all the positive integers from 10 to 99, inclusive. She next adds together all the numbers ending with a 3, 1, 4, 5, 9, and crosses them off her list. She then adds together all the remaining numbers. How much larger is the second total than the first?

8. For a given number $b > 0$, draw lines $2y = x + b$ and $2y = x + b + 10$ and $x = 6$. Together with x and y axes, these lines from two non-overlapping regions in the first quadrant. For which value of b do the regions have the equal area?

9. Given that `A, B, C, D, E, F, G, H, I` are distinct nonzero digits such that the product of `A` and `B` is equal to (two-digit) number `CD`, and the product of `E` and (two-digit) number `FG` is equal to (two-digit) number `HI`.

   (a) There is a very special digit, denoted by d, we should consider in solving this problem. Which digit is d? Why?

(b) Why any one of `A, B, D, E, G, I` cannot equal to d?

(c) Determine if `F` can equal to d.

(d) Find the (two-digit) number `HI`.

10. Find the largest positive integer n less than 50 such that the least common multiple of numbers $n, n+1, \ldots, 50$ is equal to that of numbers $1, 2, \ldots, 50$.

1.3 Practice tests (part 3)

1.3.1 Selected entry level algebra problems from AIME (part 3)

1. Let x, y, z all exceed 1 and let $w > 0$ with $\log_x w = 24$, $\log_y w = 40$, and $\log_{xyz} w = 12$. Find $\log_z w$.

2. Compute $\sqrt{(2012)(2014)(2016)(2018) + 16}$.

3. During a recent campaign for office, a candidate made a tour of a country which we assume lies in a plane. On the first day of the tour he went east, on the second day he went north, on the third day west, on the fourth day south, on the fifth day east, etc. If the candidate went $n^2/2$ miles on the n^{th} day of this tour, how many miles was he from his starting point at the end of the 40^{th} day?

4. Find the positive integer n for which

$$\lfloor \log_2 1 \rfloor + \lfloor \log_2 2 \rfloor + \lfloor \log_2 3 \rfloor + \cdots + \lfloor \log_2 n \rfloor = 1994.$$

(For real number x, $\lfloor x \rfloor$ denote the greatest integer less than or equal to x.)

5. Find the area of the region enclosed by the graph of

$$|x - 60| + |y| = \left| \frac{x}{4} \right|.$$

1.3.2 Selected entry level problems from Mandelbrot 2013 (part 2)

6. In a certain math class, two-thirds of the students are girls. Suppose that one-quarter of the girls are asleep, while two-fifths of the boys are asleep. What fraction of the class is awake?

7. A rim of 16 unit squares are formed by removing the 9 central unit squares from a 5×5 unit-square grid. Suppose we place thirteen chips on the squares so that each line contains exactly five chips. (It is OK to place more than one chip per square.) How many chips must be placed on the four corner squares in total?

8. A square is sliced into 1000 congruent rectangles using 999 horizontal lines. If each rectangle has a perimeter of 2013, then what is the perimeter of the original square, rounded to the nearest hundred?

9. Solve

$$\frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{x+5}.$$

10. Find all pairs (a, b) of natural numbers with $a < b$ having the property that there exists a right triangle with legs of length a and b whose hypotenuse has length $\frac{ab}{3} - a - b$.

1.4 Practice tests (part 4)

1.4.1 Selected entry level algebra problems from AIME (part 4)

1. Determine the value of ab if $\log_8 a + \log_4 b^2 = 5$ and $\log_8 b + \log_4 a^2 = 7$.

2. The function f has the property that, for each real number x,

$$f(x) + f(x-1) = x^2.$$

If $f(19) = 94$, what is the remainder when $f(94)$ is divided by 1000?

3. Suppose that r is a real number for which

$$\left\lfloor r + \frac{19}{100} \right\rfloor + \left\lfloor r + \frac{20}{100} \right\rfloor + \cdots + \left\lfloor r + \frac{91}{100} \right\rfloor = 546.$$

Find $\lfloor 100r \rfloor$.

4. Determine $3x_4 + 2x_5$, if x_1, x_2, x_3, x_4, x_5 satisfy the system of equations:

$$\begin{aligned} 2x_1 + x_2 + x_3 + x_4 + x_5 &= 6, \\ x_1 + 2x_2 + x_3 + x_4 + x_5 &= 12, \\ x_1 + x_2 + 2x_3 + x_4 + x_5 &= 24, \\ x_1 + x_2 + x_3 + 2x_4 + x_5 &= 48, \\ x_1 + x_2 + x_3 + x_4 + 2x_5 &= 96. \end{aligned}$$

5. Find the positive solution to

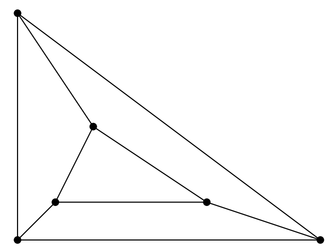
$$\frac{1}{x^2 - 10x - 29} + \frac{1}{x^2 - 10x - 45} - \frac{2}{x^2 - 10x - 69} = 0.$$

1.4.2 Selected entry level problems from Mandelbrot 2013 (part 3)

6. A certain sequence of 99 numbers begins with 100 and ends with 2014. If each term of the sequence after the first is larger than the previous term by the same amount, then what is the value of the middle term?

7. A pair of concentric circles create a ring-shaped region with shaded area 20π and total boundary (counting both the inside and outside circles) of 14π. What is the width of the ring?

8. Stuart asks Shivani, "What is your favorite integer?" She replies, "If you multiply one more than my favorite integer by 14 and subtract our classroom number, you get the square of my integer." If this is enough information for Stuart to deduce Shivani's integer, then what is their classroom number?



9. In the diagram shown on the right-hand side, how many ways are there to color two of the dots red, two of the dots blue, and two of the dots green so that dots of the same color are joined by a segment?

10. At a certain party there are seven individuals who each know precisely seven other people at the party, while the remaining guests each know exactly five other people.
    (Knowing is mutual, meaning that if one person knows another, then that second person also knows the first.)

    (a) Is it possible that the total number of guests at the party is 8?

    (b) If a guest knows exactly k other guests, then we say this guest has a *popularity index* of k. Consider the sum of all of the popularity indices of all the people at the party. Can this sum be an odd number?

    (c) Is it possible that the total number of guests at the party is 9?

    (c) Determine the minimum number of people that could be at the party?

1.5 Practice tests (part 5)

1.5.1 Selected medium to challenging level problems from AMC 2013 (part 2)

1. A basketball teams players were successful on 50% of their two-point shots and 40% of their three-point shots, which resulted in 54 points. They attempted 50% more two-point shots than three-point shots. How many three-point shots did they attempt?

2. In triangle ABC, medians AD and CE intersect at P. Given that $PE = 3/2$, $PD = 2$, and $DE = 5/2$, compute the area of quadrilateral $AEDC$.

3. The real numbers c, b, a form an arithmetic sequence with $a \geq b \geq c \geq 0$. The quadratic $ax^2 + bx + c$ has exactly one root. What is this root?

4. In base 10, the number 2013 ends in the digit 3. In base 9, on the other hand, the same number is written as $(2676)_9$ and ends in the digit 6. For how many positive integers b does the base-b representation of 2013 end in the digit 3?

5. The regular octagon $ABCDEFGH$ has its center at J. Each of the vertices and the center are to be associated with one of the digits 1 through 9, with each digit used once, in such a way that the sums of the numbers on the lines AJE, BJF, CJG, and DJH are equal. In how many ways can this be done?

1.5.2 Selected medium to challening level problems from AMC 2013 (part 3)

6. Define $a♣b = a^2bab^2$. Describes the set of points (x, y) for which $x♣y = y♣x$?

7. Alex has 75 red tokens and 75 blue tokens. There is a booth where Alex can give two red tokens and receive in return a silver token and a blue token, and another booth where Alex can give three blue tokens and receive in return a silver token and a red token. Alex continues to exchange tokens until no more exchanges are possible. How many silver tokens will Alex have at the end?

8. Two non-decreasing sequences of nonnegative integers have different first terms. Each sequence has the property that each term beginning with the third is the sum of the previous two terms, and the seventh term of each sequence is N. What is the smallest possible value of N?

9. A positive integer n is nice if there is a positive integer m with exactly four positive divisors (including 1 and m) such that the sum of the four divisors is equal to n. How many numbers in the set $\{2010, 2011, 2012, \ldots, 2019\}$ are nice?

10. Two bees start at the same spot and fly at the same rate in the following directions. Bee A travels 1 foot north, then 1 foot east, then 1 foot upwards, and then continues to repeat this pattern. Bee B travels 1 foot south, then 1 foot west, and then continues to repeat this pattern. In what directions are the bees traveling when they are exactly 10 feet away from each other?