1.3 Number sense (part 1)

1. Show that $p = 3$ is the only prime such that $p^{1994} + p^{1995}$ is a perfect square.

2. Positive integer k divides $2n^3 + 3n^2 + n$ for every positive integer n. Determine with justification the maximum value of k?

3. What is the first time after 4:56 (AM) when the 24-hour expression for the time has three consecutive digits that form an increasing arithmetic sequence with difference 1? (For example, 23:41 is one of those moments, while 23:12 is not.)

4. Let
$$S_n = \lfloor\sqrt{1}\rfloor + \lfloor\sqrt{2}\rfloor + \lfloor\sqrt{3}\rfloor + \cdots + \lfloor\sqrt{n}\rfloor.$$
Find the greatest integer k such that $S_{1997} - S_k$ is the square of a positive integer. (For real number x, $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x.)

5. From the list of integers $1, 2, 3, \ldots, 30$, Jordan can pick at least one pair of distinct numbers such that none of the 28 other numbers are equal to the sum or the difference of this pair. Of all possible such pairs, Jordan chooses the pair with the least sum. Which two numbers does Jordan pick?

6. For a pair of (positive) integers m and n, recall that their *greatest common divisor*, denoted by $\gcd(m, n)$), is the greatest integer divides both of them. Consider the following statement.

   If m and n are integers, then
   $$\gcd(m, n) = \gcd(m, m + n) = \gcd(m, m - n) = \gcd(n, m + n) = \gcd(n, m - n)$$

   Understand this statement by checking at least five numerical examples that no one else in the class will think of. Show that this statement is true.

7. (Continuation) Apply the above result effectively (and repeatedly) to compute each of the following.

   (a) $\gcd(1081, 1403)$ (b) $\gcd(4476, 13801)$
   (c) $\gcd(20259, 26160)$ (d) $\gcd(11^{13} + 13^{11}, 11^{13} - 13^{11})$

   The procedure you developed is called the *Euclidean algorithm*. Explain why it is much more effective to apply the Euclidean algorithm rather than factoring in finding the greatest common divisors of pairs of integers.

8. Let n be a five-digit number, and let q and r be the quotient and remainder, respectively, when n is divided by 100. For how many values of n is $q + r$ divisible by 11?

9. How many positive integers have exactly three proper divisors, each of which is less than 50? (A *proper divisor* of a positive integer n is a positive integer divisor other than n.)

10. Determine the remainder when

$$2^1 + 2^{1+2} + 2^{1+2+3} + \cdots + 2^{1+2+\cdots 2011}$$

is divided by 7.