## 1.11 The circumcenter and the centroid of a triangle

1. Explain why the three medians cut the triangle into 6 smaller triangles with equal area.

2. Prove the following statements.

   (a) A quadrilateral is cyclic if and only if its opposite angles are supplementary to each other.

   (b) A parallelogram is cyclic if and only if it is a rectangle.

   (c) A trapezoid is cyclic if and only it is isosceles.

   (d) If a kite is cyclic, then one of its diagonals is a diameter of the circumcircle of the kite. Furthermore, the endpoints of this diagonal are the midpoints of the minor and major arcs (of the circumcircle) formed by the two other vertices of the kite.

3. Draw a triangle $ABC$ and two of its medians $AM$ and $BN$. Let $G$ be the point where $AM$ intersects $BN$. Extend $AM$ to the point $P$ such that makes $GM = MP$. Extend $BN$ to the point $Q$ such that makes $GN = NQ$.

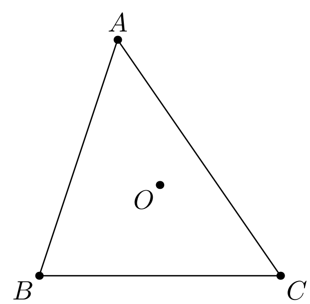
   (a) Explain why $BG$ must be parallel to $PC$, and $AG$ must be parallel to $QC$.

   (b) What kind of a quadrilateral is $PCQG$? How do you know?

   (b) Find two segments in your diagram that *must* have the same length as $BG$.

   (c) How do the lengths of segments $BG$ and $GN$ compare?

   (d) How does this result lead to another proof of the existence of the centroid of a triangle?




4. Quadrilateral $ABCD$ is inscribed in a unit circle. If $BD$ is the diameter of the circle, find the distance between the centroids of triangles $ABC$ and $ADC$.

5. An acute triangle $ABC$ with its circumcenter $O$ is shown in the diagram. Explain how to construct a triangle with angles $180° - 2A$, $180° - 2B$, and $180° - 2C$, using only a pencil and a rectangular sheet of paper. (Indeed, it can be done in two different ways.)

## 1.18 Producing similar triangles

1. Two flagpoles stand 21 and 28 feet tall. Two ropes connect the top of each flagpole to the base of the other flagpole. A bird is perched where the two ropes intersect. How far above the ground is the bird, in feet? What if the two flagpoles stand $a$ and $b$ feet tall instead?

2. A triangle has two $k$-inch sides that make a 36-degree angle, and the third side is one inch long. Draw the bisector of one of the other angles. How long is it? There are several ways to calculate the number $k$. Apply at least two of them.

3. Octagon $ABCDEFGH$ is equiangular. Given that $AB = 1$, $BC = 2$, $CD = 3$, $DE = 4$, and $EF = FG = 2$, compute the perimeter of the octagon.

4. Let $ABC$ be a triangle. Points $E$ and $F$ lie on sides $AC$ and $AB$ respectively. Rays $FE$ and $BC$ meet in $D$. Point $X$ lies on segment $ED$ such that $CX \parallel AB$. Given that $AF = 2$, $FB = 3$, $BC = 4$, $CD = 5$, find $XC$, $AE/EC$, $DX : XE : EF$.

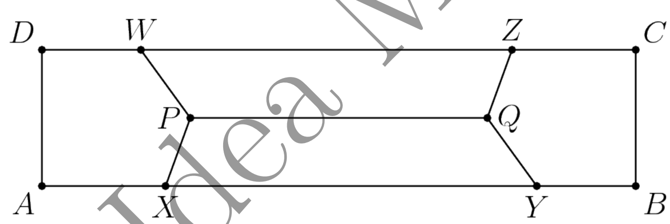
5. Let $ABC$ be a triangle. Points $E$ and $F$ lie on sides $AC$ and $AB$ respectively. Rays $FE$ and $BC$ meet in $D$. Given that $AF = 1$, $FB = 3$, $BC = 5$, $CD = 7$, find $AE/EC$ and $DE/EF$.

## 3.5 Geometry problems from AMC10/12 and AIME (part 1)

1. In parallelogram $ABCD$, point $M$ is on side $AB$ so that $AM/AB = 17/1000$, and point $N$ is on side $AD$ so that $AN/AD = 17/2009$. Let $P$ be the point of intersection of segments $AC$ and $MN$. Find $AC/AP$?

2. Square $ABCD$ has side length 2. A semicircle with diameter $AB$ is constructed inside the square, and the tangent to the semicircle from $C$ intersects side $AD$ at $E$. What is the length of segment $CE$?

3. Let $ABCD$ be a rectangle. Points $X$ and $Y$ lie on side $AB$, points $Z$ and $W$ lie on side $CD$, and points $P$ and $Q$ lie inside the rectangle, such that $XYQP$ and $PQZW$ are trapezoids. Suppose that pentagons $AXPWD$ and $BYQZC$ and trapezoids $XYQP$ and $PQZW$ have the same area, and that $XY = YB + BC + CZ = ZW = WD + DA + AX$. Find $AB$ if $BC = 19$ and $PQ = 87$.




4. Point $B$ is in the exterior of the regular $n$-sided polygon $A_1A_2\dots A_n$, and $A_1A_2B$ is an equilateral triangle. What is the largest value of $n$ for which $A_n$, $A_1$, and $B$ are consecutive vertices of a regular polygon?

5. Points $A$ and $C$ lie on circle of radius $\sqrt{50}$, and point $B$ lies inside the circle with $\angle ABC = 90°$. If $AB = 6$ and $BC = 2$, compute the distance from $B$ to the center of the circle.

## 4.3 Folding and unfolding (part 2)

1. A cylindrical tank with radius 4 feet and height 9 feet is lying on its side. The tank is filled with water to a depth of 2 feet. What is the volume of the water, in cubic feet?

2. Centers of adjacent faces of a unit cube are joined to form a regular octahedron. What is the volume of this octahedron?

3. A $3 \times 4 \times 5$ rectangular box is unfolded to obtain a (planar) polygonal region. What is the minimal perimeter of the region?

4. A string is wrapped symmetrically around a cylindrical rod. It starts at point $A$ on the top rim of the rod, wraps around the rod exactly 4 times, and ends at point $B$ on the bottom rim of the rod, where segment $AB$ is parallel to the symmetry axis of the rod. The circumference of the rod is 4 and it height is 12. What is the length of the string?

5. In the diagram shown on the right, a spider lived in a room that measured 30 feet long by 12 feet wide by 12 feet high. One day, the spider spied an incapacitated fly across the room, and of course wanted to crawl to it as quickly as possible. The spider was on an end wall, one foot from the ceiling and six feet from each of the long walls. The fly was stuck one foot from the floor on the opposite wall, also midway between the two long walls. Knowing some geometry, the spider cleverly took the shortest possible route to the fly and ate it for lunch. How far did the spider crawl?