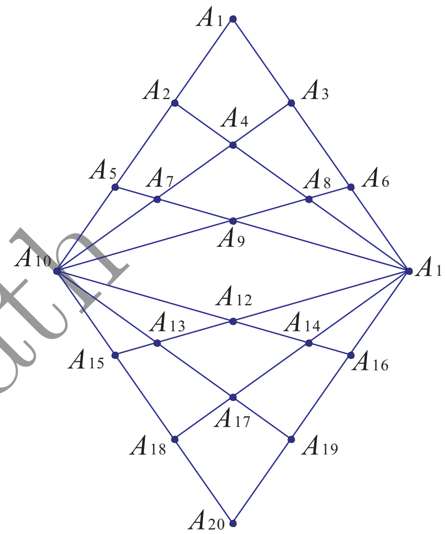
1.6 Counting practice (part 4)



1. In how many ways can Mr. Fat travel from A_1 to A_{20} along the paths in the figure shown on the right-hand side, assuming that he needs to go downward all of the time?

2. For a permutation $p = (a_1, a_2, \ldots, a_9)$ of $(1, 2, \ldots, 9)$, let $n(p)$ denote the maximum of the three products $a_1a_2a_3, a_4a_5a_6, a_7a_8a_9$, and let m denote the minimum value of $n(p)$ for all possible permutations p. Determine the number of permutations p with $n(p) = m$.

3. Alex, Mel, and Chelsea play a game that has 6 rounds. In each round there is a single winner, and the outcomes of the rounds are independent. For each round the probability that Alex wins is $\frac{1}{2}$, and Mel is twice as likely to win as Chelsea. What is the probability that Alex wins three rounds, Mel wins two rounds, and Chelsea wins one round?

4. A hat contains four cards, three of which are black on one side and red on the other, while the fourth is red on both sides. Nick chooses a card at random, looks at only one side, and observes that it is red. What is the probability that he has chosen the card that is red on both sides?

5. Determine the number of permutations of the 15-character string `AAAABBBBBCCCCCC` such that

   (a) None of the first four letters is an `A`;

   (b) None of the next five letters is a `B`;

   (c) None of the last six letters is a `C`.

1.7 Number of subsets

1. Ten points are marked on a circle. How many distinct convex polygons of three or more sides can be drawn using some (or all) of the ten points as vertices? (Polygons are distinct unless they have exactly the same vertices.)

2. Let S be a set with six elements. In how many different ways can one select two not necessarily distinct subsets of S so that the union of the two subsets is S? The order of the selection does not matter; for example the pair of subsets $\{a, c\}$, $\{b, c, d, e, f\}$ represents the same selection as the pair $\{b, c, d, e, f\}$, $\{a, c\}$.

3. Let S be a set with seven elements. Find the number of unordered pairs (X, Y) of subsets of S so that the intersection of X and Y contains an odd number of elements.

4. For any set S, let $|S|$ denote the number of elements in S, and let $n(S)$ be the number of subsets of S, including the empty set and S itself. If A, B, and C are sets for which

$$n(A) + n(B) + n(C) = n(A \cup B \cup C) \quad \text{and} \quad |A| = |B| = 100,$$

then what is the minimum possible value of $|A \cap B \cap C|$?

5. Let set $S = \{1, 2, 3, 4, 5, 6\}$, and let set T be the set of all subsets of S (including the empty set and S itself). Let t_1, t_2, t_3 be elements of T, not necessarily distinct. The ordered triple (t_1, t_2, t_3) is called *satisfactory* if either

   (a) both $t_1 \subseteq t_3$ and $t_2 \subseteq t_3$, or

   (b) both $t_3 \subseteq t_1$ and $t_3 \subseteq t_2$.

   Compute the number of satisfactory ordered triples (t_1, t_2, t_3).