Lectures on Challenging Mathematics

Core Computational Mathematics
Volume 1.4

UC1 Number Sense

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"Cogito ergo Sum" – "I think, therefore I am"

René Descartes (1596-1650)

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1.5 Simon's favorite factoring trick

1. There are two (main) approaches to determine all pairs (m, n) of integers such that

$$\frac{1}{m} - \frac{1}{n} = \frac{1}{6}.$$

We rewrite the equation as $6n - 6m = mn$ or $mn + 6m - 6n = 0$. We can either *factor* the last equation as $(m-6)(n+6) = -36$ or solve the last equation for m in terms of n (or, similarly, for n in terms of n) as

$$m = \frac{6n}{n+6}.$$

Complete the work for both approaches. (In the second approach, one needs to convert the improper fraction $\frac{6n}{n+6}$ into the mixed form.)

2. Find all pairs (m, n) of integers satisfying the equation

(a) $\frac{1}{m} + \frac{1}{n} = \frac{1}{4}$ (b) $\frac{1}{m} - \frac{1}{n} = \frac{1}{4}$

3. For each of the following equations, determine the number of its solutions in integers and find these solutions.

(a) $\frac{2}{m} - \frac{1}{n} = \frac{4}{7}$ (b) $\frac{3}{m} + \frac{1}{n} = \frac{2}{5}$

4. Three non-overlapping regular plane polygons all have sides of length 1. The polygons meet at a point A in such a way that the sum of the three interior angles at A is $360°$. Thus the three polygons form a new polygon $\mathcal{P}$ (not necessarily convex) with A as an interior point. One of the three polygons is a square. Find the possible values of the perimeter of $\mathcal{P}$.

5. Find all triples of positive integers (a, b, c) such that three rectangular bricks $a \times b \times 1$, $b \times c \times 1$, and $c \times a \times 1$ can be assembled into one $a \times b \times c$ brick.

1.6 Playing with numbers (part 1)

1. Let k be the smallest of six consecutive positive integers. If the sum of the six integers is divisible by three distinct primes, compute the smallest possible value for k.

2. Find the largest prime dividing $98^3 + 99^3 + 100^3 + 101^3 + 102^3$.

3. Find all positive integers n with $n \leq 2014$ such that $\dfrac{n!(n+1)!}{20+14}$ is a perfect square.

4. Find the greatest integer k such that 343^k divides $343!$.

5. Determine if there are infinitely many positive integers n such that the sum of the digits in the decimal representation of n^2 is equal to that of $(n+1)^2$.