Lectures on Challenging Mathematics

Core Computational Mathematics
Volume 1.3

UC1 Geometry

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"Cogito ergo Sum" – "I think, therefore I am"

René Descartes (1596-1650)

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1.3 Sentry theorem (part 1)

1. The sides of an equilateral triangle are 12 cm long. How long is an altitude of this triangle? What are the angles of a right triangle created by drawing an altitude? How does the short side of this right triangle compare with the other two sides?

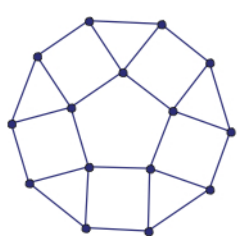
2. A trapezoid has a 60-degree angle.

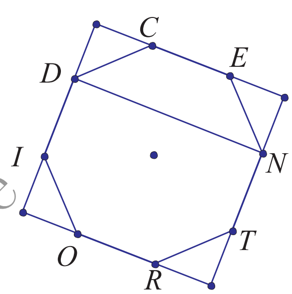
   (a) If it also has a 45-degree angle. What are the other angles?

   (b) If it also has a 120-degree angle. What are the other angles?

3. In triangle ABC, $AB = AC$. Point D lies on side AC such that $AD = DB = BC$. Compute the angles of triangle ABC.

4. Suppose that $ANGEL$ is a regular pentagon, and that $CANT$, $HALF$, $ROLE$, $KEGS$, and $PING$ are squares attached to the outside of the pentagon. Show that decagon $PITCHFORKS$ is equiangular. Explain that this decagon is *not* equilateral and hence *not* regular. If we start with a regular polygon other than a regular pentagon, can we obtain a regular polygon with the same construction process?





5. A stop sign — a regular octagon — can be formed from a 12-inch square sheet of metal by making four straight cuts that snip off the corners. How long are the sides of the resulting polygon?

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1.11 Why is it important to sketch accurate diagrams?

1. A *bisector* of an angle is a ray that divides the angle into two congruent and adjacent angles. Draw two intersecting line ℓ_1 and ℓ_2 on a piece of paper. These two lines form four (non overlapping) angles.

   (a) Explain that one can draw two lines each of which bisects two of the four angles.

   (b) It is possible to fold the paper properly to form these two bisectors. Do so.

   (c) It seems that these two bisectors form a special angle. Is it? If so, what is the measure of this angle?

2. A *median* of a triangle is a segment from a vertex to the midpoint of the opposite side.

   Draw a large triangle on a piece of paper, and label the vertices A, B, C. It is possible to fold the paper properly so we can locate the midpoint M of side BC and thus draw median AM. Do so. Also, fold the paper properly to obtain the (interior) angle bisector of angle A and the altitude from A.

3. In triangle ABC, assume that the interior bisector of angle A meet the perpendicular bisector of side BC in point P that lies in the interior of triangle ABC. Points D and E lie on sides AC and AB respectively such that $PE \perp AB$ and $PD \perp AC$.

   (a) Prove that triangles ADP and AEP are congruent by AAS.

   (b) Prove that triangles BEP and CDP are congruent.

   (c) Conclude that $AB = AC$. Hmm ... Something must be wrong, is it?

   To solve the puzzle, draw an accurate diagram of the following: A scalene triangle ABC, with its altitude, median, interior and exterior bisectors at vertex A, and perpendicular bisector of side BC. Observe the relative positions of these five lines and compare your observations with your peers. What if two of these five lines (or segments) coincide with each other?

4. In triangle ABC, point N lies on side AC such that $AN = BN = CN$. Segment BF is the altitude drawn from B to AC. Find the angles ANB and NBF, given that

   (a) angle C is $43°$ degrees; (b) angle C is $47°$ degrees.

5. Jane is solving the following problem:

   > Let ABC be a triangle with $AB = 8$, $AC = 9$, and $BC = 4$. Find the length of the altitude AH.

   In her diagram, she draws point H on segment BC and denotes $BH = x$ and $CH = 4 - x$. Then she expresses AH in terms of x in two ways and solves for x.

   (a) Use Jane's approach to find x. Something must be wrong, is it?

   (b) Fix Jane's approach and find the length of AH.

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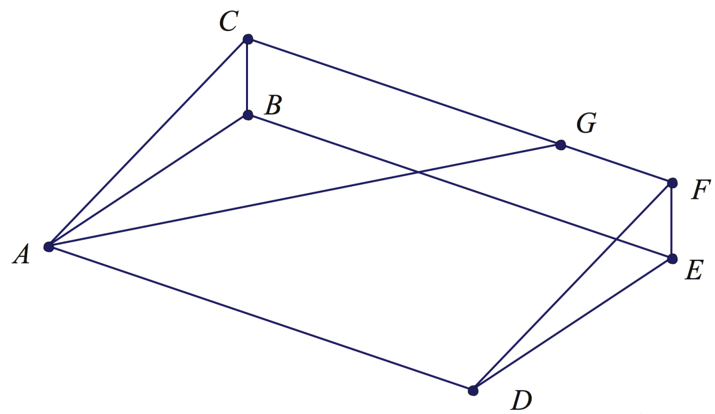
1.19 Pythagorean theorem (part 3)

1. Two angles of a triangle measure 30 and 45 degrees. If the side of the triangle opposite the 30-degree angle measures 12 units, what is the sum of the lengths of the two remaining sides?

2. In triangle ABC, $\angle C = 90°$. Medians are drawn from point A and point B in this right triangle to divide segment BC and AC in half, respectively. The lengths of the medians are 6 and $2\sqrt{11}$ units, respectively. How many units are in the length of segment AB?

3. Let DEF be a triangle and H the foot of the altitude from D to EF. If $DE = 60$, $DF = 35$, and $DH = 21$, what is the difference between the minimum and the maximum possible values for the area of DEF?

4. Peyton's workout today is to run repeatedly up a steep grassy slope, represented by $ADFC$ in the diagram. The workout loop is $AGCA$, in which AG requires exertion and GCA is for recovery. Point G was chosen on the ridge CF to make the slope of the climb equal 20%.



Given that $ADEB$ and $BEFC$ are rectangles, ABC is a right angle, $AD = 240$, $DE = 150$, and $EF = 50$, find the distance from point G to point C.

5. (Continuation) Peyton's next workout loop is $AHCA$, where H is a point on the path AG, chosen to make the slope of HC equal 20%. Find the ratio AH/AG, and explain your choice.

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1.20 A project on coordinate geometry

1. In the coordinate plane, $P = (x, y)$ and $C = (3, 5)$. Write a mathematical expression to describe the property that P is 5 units away from C. What is the common mathematical term describing the *locus* of P. What is the common term describing point C in relation to this familiar locus?

2. In the coordinate plane, find the *locus* of P for each of the following.

   (a) $x^2 + y^2 = 9$ (b) $(x - 3)^2 + (y - 5)^2 = 25$

   (c) $(3 - x)^2 + (y + 4)^2 = 49$ (d) $(x - 3)^2 + (5 - y)^2 = 81$

3. The graph of each of the equations below is a circle. For every equation locate the center of the circle and find its radius.

   (a) $x^2 + y^2 - 6x + y = 3$ (b) $x^2 + y^2 + 8x = 0$

   (c) $x^2 + y^2 + 2x - 8y = -8$ (d) $10x^2 + 10y^2 = 4x + 65y$

4. Find the intersections of the circles $x^2 + y^2 = 100$ and $(x - 21)^2 + y^2 = 289$.

5. (Continuation) Find an equation of the line passing through the two intersections. Simplify the expression $x^2 + y^2 - 100 = (x - 21)^2 + y^2 - 289$. Hmm ... .

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2.4 Revisiting special quadrilaterals

1. In square $ABCD$, points M, N, P are midpoints of segments AB, AD, MN, respectively. Point Q lies on segment BD such that quadrilateral $PQDN$ is a parallelogram. What is the ratio of the area of parallelogram $PQDN$ to the area of square $ABCD$?

2. In rectangle $ABCD$, points E, F, G, H lie on sides AB, BC, CD, DA, respectively, so that $EFGH$ is a parallelogram. Given that $AH = CF = 3$, $AE = CG = 4$, $BF = DH = 5$, and $BE = DG = 6$, find the distance between sides HE and FG.

3. A kite has a 5-inch side and a 7-inch side. One of the diagonals is bisected by the other. The bisecting diagonal has length 8 inches. Find the length of the bisected diagonal.

4. Let $ABCD$ be a quadrilateral such that $[ABC] = [BCD]$. Prove that diagonal BC divides diagonal AD in half.

5. Jordan wants to create an equiangular octagon whose side lengths are exactly the first 8 positive integers, so that each side has a different length. How many such octagons can Jordan create?

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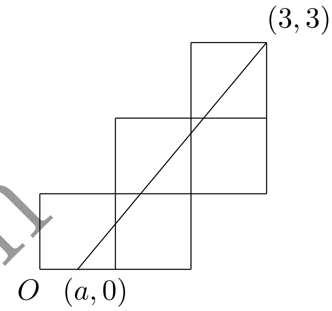
2.9 Practices in geometric computations (part 5)

1. In triangle ABC, $AB = AC$. Points D and E lie on sides AB and AC respectively such that $AD = DE = EB = BC$. Find $\angle A$.

2. Five unit squares are arranged in the coordinate plane as shown, with the lower left corner at the origin O. The slanted line, extended from $(a, 0)$ to $(3, 3)$, divides the entire region into two regions of equal area. What is a?



3. Let $ABCD$ be a trapezoid with $AB \parallel CD$. Denote by M and N the midpoints of AB and CD. Suppose that $AD = 4$, $AB = 7$, $BC = 3$, and $CD = 12$. Find the length of the segment MN.

4. Consider a rectangle $ABCD$ with side lengths $AB = CD = 6$ and $BC = DA = 8$. Let X be a point on BD such that $AX \perp BD$. Find the length of each of following segments:

   (a) AX (b) BX (c) CX

5. Let $ABCD$ be a square. Points P and Q are variable points on segments AB and CD respectively such that the ratio between the areas of quadrilaterals $APQD$ and $BPQC$ is equal to $2 : 3$. Show that line PQ passes through a fixed point.

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