Lectures on Challenging Mathematics

Core Computational Mathematics
Volume 1.2

UC1 Counting

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"Cogito ergo Sum" – "I think, therefore I am"

René Descartes (1596-1650)

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1.7 Simple counting practices (part 5)

1. The digits of the number

$$N = 12345678910111213\ldots 20092010$$

are the integers 1 through 2010 in order from left to right. What is the remainder N divided by 3?

2. A game uses a deck that consists of n different cards, where n is an integer and $n \geq 6$. The number of possible sets of six cards that can be drawn from the deck is 6 times the number of possible sets of three cards that can be drawn. Find n.

3. Let n be the smallest positive integer that is a multiple of 75 and has exactly 75 positive integral divisors, including 1 and itself. Find $n/75$.

4. Let a and b be digits from the set $\{0, 1, 2, \ldots, 9\}$. Let r be the two-digit string $\overline{ab}$ and let s be the two-digit string $\overline{ba}$, so that r and s are members of the set $\{00, 01, \ldots, 99\}$. Compute the number of ordered pairs (a, b) such that $|r - s| = k^2$ for some integer k.

5. Let M be the number of non-empty subsets of $\{2, \ldots, 50\}$ that do not contain any number that is divisible by the square of a prime. Find the greatest prime divisor of M.

1.9 Simple counting practices (part 9)

1. Alex is given a number. Every second, Alex can add or subtract any number of the form $n!$ to his current number to get a new number. In how many ways can Alex get from 0 to 100 in 4 seconds?

2. Six people, all of different weights, are trying to build a human pyramid: that is, they get into the formation shown on the right. We say that someone not in the bottom row is *supported by* each of the two closest people beneath her or him. How many different pyramids are possible, if nobody can be supported by anybody of lower weight?

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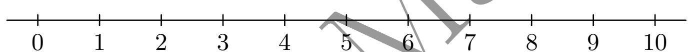
  A
 B C
D E F
```

3. Eleven marks are shown on a regular ruler. Seven marks are to be erased. If we want to be able to measure all integer distances up to n. What is the maximum number of n?



4. Using each letter once from the set `S, I, M, P, L, E` we form distinct six-letter codes. If these codes are arranged in alphabetical order, then what position does the code `SIMPLE` occupy?

5. Determine the number of positive integer divisors of the product $3! \cdot 5! \cdot 7! \cdot 9! \cdot 11!$ that are perfect squares.

1.13 Counting practices (part 3)

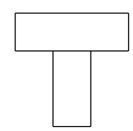
1. Five lines and two circles with different radii are drawn in the plane. What is the maximum number of possible intersection points among these seven figures?

2. We are trying to give Alice, Bob, Carla, and Daniel seven (indistinguishable) pieces of chocolate chip cookies in such a way that each of them has at least one piece of cookie. How many ways are there to do this?

3. A rectangular track is a region consisting of 1×1 squares arranged in a rectangle shape. For example, a 3×6 rectangular track contains 14 squares, and a 5×8 rectangular track contains 22 squares. What is the total number of squares in a 3×6, a 5×8, a 7×10, $\ldots$, and a 97×100 rectangular track?

4. Find the smallest positive integer m that has at least 21 positive integer factors. Determine the product of distinct positive integer divisors of m.

5. In how many ways can Alex draw the diagram below without lifting his pencil or retracing a line? Two drawings are different if the order in which he draws the edges is different, or the direction in which he draws an edge is different.



1.18 Counting practices (part 8)

1. A year is a leap year if and only if the year number is divisible by 400 (such as 2000) or is divisible by 4 but not by 100 (such as 2012). The 200th anniversary of the birth of novelist Charles Dickens was celebrated on February 7, 2012, a Tuesday. On what day of the week was Dickens born?

2. How many distinct non-regular polygons can be formed by connecting some different vertices of a regular dodecagon $A_1A_2\dots A_{12}$?

3. Zeb has twenty $1 \times 2 \times \pi$ bricks. He stacks them, one on top of another, to form a tower twenty bricks high. Each brick can be in any orientation so long as it rests on top of the next brick below it (or on the floor). How many distinct heights of towers can he make?

4. Some students in a gym class are wearing blue jerseys, and the rest are wearing red jerseys. There are exactly 25 ways to pick a team of three players that includes at least one player wearing each color. Compute the number of students in the class.

5. If a, b, c, d are one-digit positive integers, determine how many ordered quadruples (a, b, c, d) are there which satisfy $a + bcd = ab + cd$.