Lectures on Challenging Mathematics

Core Computational Mathematics
Volume 1.1

UC1 Algebra

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"Cogito ergo Sum" – "I think, therefore I am"

René Descartes (1596-1650)

Contents

1.7 Setting up (quadratic) equations (part 1)

(Many of the problems in this section are selected from PEA Math 1 materials.)

1. Two automobiles each travel 60 km at steady rates. One car goes 6 kph faster than the other, thereby taking 20 minutes less time for the trip. Find the rate of the slower car.

2. A debt of \$450 is to be shared equally among the members of the Outing Club. When five of the members refuse to pay, the other members' shares each go up by \$3. How many members does the Outing Club have?

3. A rectangular box has length x, width $x - 2$, and height $2x + 1$. If its total surface is 734, find the volume of the box.

4. The diagram at right shows the flag of Sweden, which consists of a gold cross of uniform width against a solid blue background. The flag measures 3 feet 4 inches by 5 feet 4 inches, and the area of the gold cross is 30% of the area of the whole flag. Use this information to find the width of the gold cross.

5. Mr. Atf has 5040 candy bars and a pack of paper bags. He can distribute all the candy bars evenly into all the bags. If he takes out four candy bars from each bag, he needs an extra 80 bags to distribute the candies that were taken out to obtain an even distribution. Determine, with justification, the number of bags in the pack.

1.13 Quadratic inequalities

1. Sketch the graphs of $y = 2x^2 - 6x$ and $y = -2x^2 + 7x - 3$. Solve the each of the inequalities $2x^2 - 6x \geq 0$ and $3 \geq 7x - 2x^2$.

2. When asked to solve the equation
$$\frac{x-2}{2x-3} > 0,$$
Alex announced a plan: "Just solve for the product instead." What does Alex mean? Solve this inequality.

3. Are the following pairs of inequalities equivalent to each other? If not, how do they differ from each other?

   (a) $x^2 + 6 > 5x$ and $\dfrac{x-2}{3-x} < 0$

   (b) $\dfrac{3x+2}{x-3} \leq 0$ and $(3x+2)(3-x) \geq 0$

4. When asked to solve the equation
$$\frac{x-2}{2x-3} > 1,$$
Alex again announced a plan: "Just solve for the product instead: $(x-2)(2x-3) > 1$". Blair puzzled: "Are you sure? I think you need to do some other things first." Can you solve the inequality?

5. Solve each of the following inequalities.

   (a) $2 + \dfrac{4}{x-3} \geq \dfrac{1}{x+2}$

   (b) $|x^2 - 2x - 15| \geq 8$

1.20 Algebra techniques in arithmetic (part 1)

1. When $\dfrac{998 \cdot 994}{990 \cdot 986}$ is written in lowest term, what is the positive difference between its numerator and denominator?

2. Evaluate $\dfrac{4^4 + 4^4 + 4^4 + 4^4 + 4^4}{2^6 + 2^6 + 2^6 + 2^6 + 2^6 + 2^6} \cdot \dfrac{6^2 + 6^2 + 6^2 + 6^2}{3^4 + 3^4 + 3^4}$.

3. Suppose x, y, z are real numbers that satisfy:

$$\begin{aligned} xy + yz - zx &= 10 \\ yz + zx - xy &= 12 \\ zx + xy - yz &= 14 \end{aligned}$$

   Find $x^2 + y^2 + z^2$.

4. Simplify $\sqrt{16 - 4\sqrt{15}}$ and $2\sqrt{3 + \sqrt{5 - \sqrt{13 + \sqrt{48}}}}$.

5. Find all positive integers n less than 50 such that the fraction

$$\frac{3n + 11}{n - 3}$$

   is not a fraction of the lowest form.

1.23 Quadratic curves (part 5)

(Many of the problems in this section are selected from PEA Math 1 materials.)

1. Graph the equation $y = -2x^2 + 5x + 33$. Determine all values of x for which

   (a) $y = 0$ (b) $y = 21$
   (c) $y \geq 0$ (d) $21 - y \geq 0$

2. Let x and y be real numbers such that $3x + 4y = 12$. Find the maximum possible value of xy. It might be helpful to write xy as a product in one variable. If k is the maximum, graph both equations, $xy = k$ and $3x + 4y = 12$, in the same coordinate plane.

3. Graph the nonlinear equation $y = 9 - x^2$, identifying all the axis intercepts. On the same system of coordinate axes, graph the line $y = 3x - 5$, and identify its axis intercepts. Find the coordinates of the points where the line intersects the parabola.

4. Let $\mathcal{C}$ denote the graph of $y = ax^2 + bx + c$. Given that $\mathcal{C}$ intersects the line $y = 2x + 3$ in exactly *one* place, what is the relation between a, b, and c? Provide an example of an integer triple (a, b, c) satisfying the relation and draw the graph of $\mathcal{C}$ and the line $y = 2x + 3$. What if $\mathcal{C}$ intersects the line in exactly *two* places? What if $\mathcal{C}$ does not intersect the line? Can $\mathcal{C}$ intersect the line in more than two places?

5. A trapezoid is inscribed in the parabola $y = x^2 - 3x + 1$. The bases of the trapezoid are parallel to the x-axis. The height of the trapezoid and the length of the one of the bases are both equal to 2. Find the area of the trapezoid.

2.4 Challenges with graphs (part 2)

1. Consider $y = |x+1| + |x-2|$. Explain why it is equivalent to piecewise relation

$$y = \begin{cases} -2x+1, & \text{for } x \geq -1, \\ 3, & \text{for } -1 \leq x \leq 2, \\ 2x-1, & \text{for } x \geq 2. \end{cases}$$

2. (Continuation) Consider the graph of $y = |x+1| + |x-2|$.

   (a) Plot the points on the graph with the absolute values of their x-coordinate less than 10.

   (b) Sketch the graph.

   (c) For which value(s) the function achieves its minimum?

3. Define a piece wise function with its graph match that of $y = |2|x+3| - 4|$.

4. Find all a such that $|2|x+3| - 4| = a$ has exactly n solution for $n = 0, 1, 2, 3, 4$.

5. Solve each of the following equations.

   (a) $|2|x+3| - 4| = 0$ (b) $|2|x+3| - 4| = 4$

2.9 Special equations

1. What is the sum of the three positive integers a, b, c such that
$$a + \frac{1}{b + \frac{1}{c}} = \frac{15}{4}?$$

2. The average of six distinct real number is 275. The average of the four least number is 200. The average of the four greatest number is 340. Compute average of the middle two numbers.

3. Alec, Ben, Chad, Dale, Emma, Frank, Greg, Hue, Ivy, James sit around in a circle in that order. Each person picks a number and tells it to the two neighbors adjacent to him or her in the circle. Then each person computes and announces the average of the numbers of the two neighbors. Alec, Ben, Chad, Dale, Emma, Frank, Greg, Hue, Ivy, James announce the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, respectively. What is the number picked by Frank?

4. For how many integers x is the number
$$x^4 - (51-4)(51+4)x^2 + (25-4)(51-25)(25+4)(51+25)$$
negative?

5. Find all possible values of k such that all the roots of the equation
$$(4-k)(8-k)x^2 - (80-12k)x + 32 = 0$$
are integers.