Chapter 7

Number theory challenge

7.1 Techniques in factoring

1. Factor $2^{24} - 1$ and $2^{22} + 1$.

2. Let a, b, c, d, e, f be integers (not necessarily distinct) between -100 and 100, inclusive, such that $a + b + c + d + e + f = 100$. Let M and m be the maximum and minimum possible values, respectively, of $abc + bcd + cde + def + efa + fab + ace + bdf$. Find M/m.

3. A pair of quadratic polynomials $p_1(x) = a_1x^2 + b_1x + c_1$ and $p_2(x) = a_2x^2 + b_2x + c_2$ are called *associated* if

   (i) $a_1, a_2, b_1, b_2, c_1, c_2, a_1 + a_2, b_1 + b_2, c_1 + c_2$ are nonzero integers with $\gcd(a_1, b_1, c_1) = \gcd(a_2, b_2, c_2) = 1$;

   (ii) each of $p_1(x), p_2(x)$, and their sum $p(x) = p_1(x) + p_2(x)$ can be written as the product of two linear polynomials with integer coefficients in x;

   (iii) $p_1(x)$ and $p_2(x)$ do not share a common zero.

   Determine if there are infinitely many pairs of associated quadratic polynomials.

4. The product of a 2-digit prime, a 3-digit prime, and a 4-digit prime is equal to 100140001. Find the sum of these three primes.

5. Find the least positive integer n for which there exists a set $\{s_1, s_2, \ldots, s_n\}$ of (distinct) positive integers such that

$$\left(1 - \frac{1}{s_1}\right)\left(1 - \frac{1}{s_2}\right)\cdots\left(1 - \frac{1}{s_n}\right) = \frac{51}{2010}$$