Chapter 3

Geometry knowledge

3.1 Revisit similarity (part 1)

1. A coin of radius 1 cm is tossed onto a plane surface that has been tesselated by isosceles triangles whose sides have lengths 15 cm, 15 cm, and 18 cm. What is the probability that the coin lands within one of the triangles?

2. Trapezoid $ABCD$ has $AD \parallel BC$. Diagonals AC and BD intersect at P. If $[ADP]/[BCP] = 1/2$, find $[ADP]/[ABCD]$.

3. Point P is a point inside the triangle ABC. Lines are drawn through P parallel to the sides of the triangle. The areas of the three resulting triangles with a vertex at P have areas 4, 9 and 49. What is the area of triangle ABC?

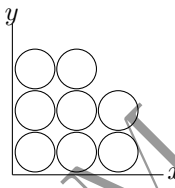
4. Triangular region $\mathcal{R}_1$ is bounded by vertices $A(0,0), B(12,12), C(12,20)$. Triangular region $\mathcal{R}_2$ is bounded by vertices $D(38,26), E(38,18), F(50,38)$. Let $\mathcal{R}$ denote the *midpoint* region of $\mathcal{R}_1$ and $\mathcal{R}_2$; that is, a point P is in $\mathcal{R}$ if and only if P is the midpoint of a pair of points Q_1 and Q_2 with Q_1 in $\mathcal{R}_1$ and Q_2 in $\mathcal{R}_2$ respectively. Find the area of $\mathcal{R}$.

5. In triangle ABC, $AB = 425$, $BC = 450$, $CA = 510$. Moreover, P is an interior point chosen so that segments DE, FG, HI are each of length d, contain P, and are parallel to sides AB, BC, CA, respectively, where D and G lie on segment CA; E and H lie on segment BC; F and I line on segment AB. Compute d.

3.2 Analytic geometry

1. What is the area of the polygon whose vertices are the points of intersection of the curves $x^2 + y^2 = 25$ and $(x-4)^2 + 9y^2 = 81$?

2. Eight circles of diameter 1 are tightly packed (tangent to each other and to the axes when positions permit) in the first quadrant of the coordinate plane as shown. Let region $\mathcal{R}$ be the union of the eight circular regions. Line ℓ, with slope 3, divides $\mathcal{R}$ into two regions of equal area. Line ℓ's equation can be expressed in the form $ax = by + c$, where a, b, and c are positive integers whose greatest common divisor is 1. Find $a^2 + b^2 + c^2$.



3. Let $A = (3, -1)$, $B = (21, 11)$, $C = (9, 29)$, $D = (-9, 17)$, $P = (1, 2)$, and $Q = (13, 23)$. A pair (S, T) of lattice points is *super* if S lies on segment AB and T on CD such that $ST = PQ$. Find two pairs of super points.

4. Starting at the origin, a beam of light hits a mirror (in the form of a line) at point $A = (4, 8)$ and is reflected to point $B = (8, 12)$. Compute the exact slope of the mirror.

5. Triangle ABC lies in the Cartesian plane and has area 70. The coordinates of B and C are $(12, 19)$ and $(23, 20)$, respectively, and the coordinates of A are (p, q). The line containing the median to side BC has slope -5. Find the largest possible value of $p + q$.

6.3 More on solid geometry (part 2)

1. A right cone is sliced by a plane that is parallel to its base, one of the two pieces is another cone; the other is a frustum. Given that a sphere can be inscribed in that frustum and that the volume of the frustum is twice that of the sphere. What is the ratio of the radius of the bottom base of the frustum to the radius of the top base of the frustum?

2. I have been observing the motion of a bug that is crawling on the sphere $x^2 + y^2 + z^2 = 361$. When I started watching, it was at the point $(1, 6, 18)$. An hour later it is at $(6, 17, -6)$.
   (a) How long is the bug's journey?
   (b) Where was the bug 20 minutes after I started watching it?

3. A solid in the shape of a right circular cone is 4 inches tall and its base has a 3-inch radius. The entire surface of the cone, including its base, is painted. A plane parallel to the base of the cone divides the cone into two solids, a smaller cone-shaped solid $\mathcal{C}$ and a frustum-shaped sold $\mathcal{F}$, in such a way that the ratio between the areas of the painted surfaces of $\mathcal{C}$ and $\mathcal{F}$ and the ratio between the volumes of $\mathcal{C}$ and $\mathcal{F}$ are both equal to k. Given that $k = m/n$, where m and n are relatively prime positive integers, find $m + n$.

4. Chad and Jordan are in the Exeter Space Station, which is a triangular prism with equilateral bases. Its height has length one decameter and its base has side lengths of three decameters. To protect their station against micrometeorites, they install a force field that contains all points that are within one decameter of any point of the surface of the station. What is the volume of the set of points within the force field and outside the station, in cubic decameters?

5. A cube with side length 10 is suspended above a plane. The vertex closest to the plane is labeled A. The three vertices adjacent to vertex A are at heights 10, 11, and 12 above the plane. The distance from vertex A to the plane can be expressed as $\frac{r-\sqrt{s}}{t}$, where r, s, and t are positive integers and r and t are relatively prime to each other. Find $r + s + t$.