2.4 Bijection

1. Determine the number of ways to choose five numbers from the first eighteen positive integers such that any two chosen numbers differ by at least 2.

2. Determine the number of subsets S of $\{1, 2, 3, \ldots, 10\}$ with the following property: there exist integers $a < b < c$ with $a \in S$, $b \notin S$, $c \in S$.

3. Every card in a deck has a picture in one shape–a circle, a square, or a triangle, which is painted in one of three colors–red, blue, or green. Furthermore, each color is applied in one of three shades–light, medium, or dark. The deck has 27 cards, with every shape–color–shade combination represented. A set of three cards from the deck is called *complementary* if all of the following statements are true:

   (a) Either each of the three cards has a different shape or all three cards have the same shape.

   (b) Either each of the three cards has a different color or all three cards have the same color.

   (c) Either each of the three cards has a different shade or all three cards have the same shade.

   How many different complementary three-card sets are there?

4. [By Weichao Wu] Let n be an integer with $n \geq 2$, and define the sequence $S = (1, 2, \ldots, n)$. A subsequence of S is called arithmetic if it has at least two terms and it is an arithmetic progression. An arithmetic subsequence is called maximal if this progression cannot be lengthened (at either ends) by the inclusion of another element of S. Determine the number of maximal arithmetic subsequences.

5. Draw n points $P_1, \ldots, P_n$ equally spaced around the circumference of a circle.

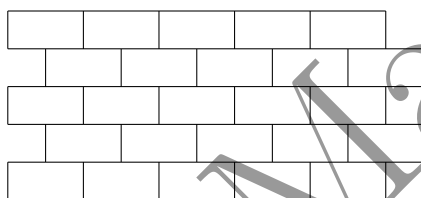
   (a) How many (unordered) triples $\{P_i, P_j, P_k\}$ are there so that triangle $P_iP_jP_k$ is acute? obtuse? Find all n such that it is equally likely for $P_jP_jP_k$ to be acute and to be obtuse.

   (c) Further assume that n is even. What is the ratio between the number of acute triangles and the number of obtuse triangles? Hmm ...

6.4 Counting practices (part 4)

1. Suppose that one of every 500 people in a certain population has a particular disease, which displays no symptoms. A blood test is available for screening for this disease. For a person who has this disease, the test always turns out positive. For a person who does not have the disease, however, there is a 2% false positive rate in other words, for such people, 98% of the time the test will turn out negative, but 2% of the time the test will turn out positive and will incorrectly indicate that the person has the disease. Let p be the probability that a person who is chosen at random from this population and gets a positive test result actually has the disease. What is p?

2. In the figure below, how many ways are there to select 5 bricks, one in each row, such that any two bricks in adjacent rows are adjacent?



3. A 3×3 square is partitioned into 9 unit squares. Each unit square is painted either white or black with each color being equally likely, chosen independently and at random. The square is then rotated 90° clockwise about its center, and every white square in a position formerly occupied by a black square is painted black. The colors of all other squares are left unchanged. What is the probability that the grid is now entirely black?

4. A frog is at the point $(0, 0)$. Every second, he can jump one unit either up or right. He can only move to points (x, y) where x and y are not both odd. How many ways can he get to the point $(8, 14)$?

5. Knot is ready to face Gammadorf in a card game. In this game, there is a deck with twenty cards numbered from 1 to 20. Each player starts with a five card hand drawn from this deck. In each round, Gammadorf plays a card in his hand, then Knot plays a card in his hand (that is, Knot sees the card that Gammadorf plays before he has to make his own move). Whoever played a card with greater value gets a point. At the end of five rounds, the player with the most points wins. If Gammadorf starts with a hand of 1, 5, 10, 15, 20, how many five-card hands of the fifteen remaining cards can Knot draw which always let Knot win (assuming he plays optimally)?