1.5 The division algorithm

1.5.1 The division algorithm

1. The following result is called the *division algorithm* and it plays an important role in number theory:

   For any positive integers a and b there exists a unique pair (q, r) of nonnegative integers such that $b = aq + r$ and $r < a$. We say that q is the *quotient* and r the *remainder* when b is divided by a.

   For a more intuitive approach of this fact: Consider the positions of b and $0, a, 2a, 3a, \ldots$ on the number line and explain the division algorithm.

2. Let N be the largest integer for which both N and $7N$ have exactly 99 digits. What is the 50^{th} leftmost digit of N?

3. Six distinct positive integers are randomly chosen between 1 and 2006, inclusive. What is the probability that some pair of these integers has a difference that is a multiple of 5? What if we choose five distinct positive integers?

4. How many integers n satisfy both of the following conditions:

   (a) $100 < n < 200$

   (b) n has the same remainder whether it is divided by 6 or by 8.

5. The sum of the respective remainders of 63, 91, 129 divided by a is equal to 25. Find a.

1.5.2 Algebra and number sense

6. Compute the number of integers n for which $2^4 < 8^n < 1632$.

7. Confirm that $(a^2 + b^2)(x^2 + y^2) = (ax + by)^2 + (ay - bx)^2$. Find another way to write $(a^2 + b^2)(x^2 + y^2)$ as the sum of two squares.

8. Factor each of the numbers 391, 899, 1591, 2491, 3599, 4209, 5561, and 5609. (A certain algebraic identity is very helpful!)

9. Writing 1313 as the sum of two perfect squares.

   (a) For digits $\texttt{A}$, $\texttt{B}$, and $\texttt{C}$, $\overline{\texttt{AB}}^2 + \overline{\texttt{AC}}^2 = 1313$. Find all possible 3-digit numbers $\overline{\texttt{ABC}}$.

   (a) For digits $\texttt{W}$, $\texttt{X}$, $\texttt{Y}$, and $\texttt{Z}$, $\overline{\texttt{WX}}^2 + \overline{\texttt{YZ}}^2 = 1313$. Find all possible 4-digit numbers $\overline{\texttt{WXYZ}}$.

   (Consider two approaches. The first is related to approximation and units digit examination. The second is related to a certain algebraic identity.)

10. Eric, Meena, and Cameron are studying the famous equation $E = mc^2$. To memorize this formula, they decide to play a game. Eric and Meena each randomly think of an integer between 1 and 50, inclusively, and substitute their numbers for E and m in the equation. Then, Cameron solves for the absolute value of c. What is the probability that Cameron's result is a rational number?

1.9 Application of divisibility rules of small numbers (part 3)

1.9.1 Application of divisibility of small numbers (part 3)

1. Is the product of all possible digits x such that the six-digit number 3414x7 is divisible by 3?

2. A positive 16-digit integer $a = \overline{a_1a_2\ldots a_{16}}$ is such that every one of $\overline{a_1a_2}, \overline{a_2a_3}, \ldots \overline{a_{15}a_{16}}$ is a 2-digit number that is a multiple of either 19 or 31. If the digit 2 appears only once in a, what is the sum of the 16 digits?

3. There exists a digit y such that, for any digit x, the seven-digit number 123x5y7 is not a multiple of 11. Compute y.

4. Let N be a six-digit number formed by an arrangement of the digits 1, 2, 3, 3, 4, 5. Compute the smallest value of N that is divisible by 264.

5. Determine the number of positive integers less than 1000 whose digits sum to a multiple of 9. How many of such numbers have the digits sum equal to 9? 18? Hmm ...

1.9.2 Multiples of small numbers (part 1)

6. A particular positive three-digit integer is divisible by 5. The integer also is divisible by 11. The sum of the three digits of the integer is 13. What is the integer?

7. Complete the following sentence:

   If the sum of three integers is divisible by 3, then the remainders of these integers divided by 3 are either ... or ... .

8. Every high school in the city of Euclid sent a team of 3 students to a math contest. Each participant in the contest received a different score. Andrea's score was the median among all students, and hers was the highest score on her team. Andrea's teammates Beth and Carla placed 37^{th} and 64^{th}, respectively. How many schools are in the city?

9. Consider the statement:

   Positive integer $n = \overline{a_ha_{h-1}\ldots a_1a_0}$ is divisible by 7 if and only if the difference $m = \overline{a_ha_{h-1}\ldots a_1} - 2 \cdot a_0$ is divisible by 7.

   Understand this statement by checking at least 5 numerical examples that no one else in the class will think of. Show that this statement is true.

10. Find the smallest positive integer n such that the decimal representation of $2010n$ has exactly one even digit.

9. Mrs. Walter gave an exam in a mathematics class of five students. She entered the scores in random order into a spreadsheet, which recalculated the class average after each score was entered. Mrs.Walter noticed that after each score was entered, the average was always an integer. The scores (listed in ascending order) were 71, 76, 80, 82, and 91. What was the last score Mrs. Walter entered?

10. The consecutive integers from 1 to n inclusive were written on a chalkboard. One of the integers was erased, and the mean of the remaining integers was $35\frac{7}{17}$. What integer was erased?