Lectures on Challenging Mathematics

Essential Computational Mathematics
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PCX Number Sense

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"Cogito ergo Sum" – "I think, therefore I am"

René Descartes (1596-1650)

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1.7 The Fundamental Theorem of Arithmetic

1. [The Fundamental Theorem of Arithmetic] Any integer n greater than 1 has a unique representation (up to a permutation) as a product of primes.

   The proof of this seemingly trivial theorem is rather difficult. For now, we just ask the reader to accept this fact. From the above theorem it follows that any integer $n > 1$ can be written uniquely in the form

   $$n = p_1^{\alpha_1} \cdots p_k^{\alpha_k},$$

   where $p_1, \ldots, p_k$ are distinct primes and $\alpha_1, \ldots, \alpha_k$ are positive integers. This representation is called the *canonical factorization* (or *factorization*) of n. It is not difficult to see that the canonical factorization of the product of two integers is the product of the canonical factorizations of the two integers. This factorization allows us to establish the following fundamental property of primes:

   Let a and b be integers. If a prime p divides ab, prove that p divides either a or b.

2. Let p, q and r be distinct prime numbers, where 1 is not considered a prime. What is the smallest positive perfect cube having $n = pq^2r^4$ as a divisor?

3. Compute the least positive odd prime p such that $p^3 + 7p^2$ is a perfect square.

4. Find all ordered pairs (m, n) of positive integers such that $m^2 - n^2 = 150$.

5. An integer is randomly chosen from $1, 2, \ldots, 100$. Determine if it is more likely to be a prime or a multiple of 4.

1.9 The division algorithm

1. The following result is called the *division algorithm* and it plays an important role in number theory:

   For any positive integers a and b there exists a unique pair (q, r) of nonnegative integers such that $b = aq + r$ and $r < a$. We say that q is the *quotient* and r the *remainder* when b is divided by a.

   For a more intuitive approach of this fact: Consider the positions of b and $0, a, 2a, 3a, \ldots$ on the number line and explain the division algorithm.

2. Let N be the largest integer for which both N and $7N$ have exactly 99 digits. What is the 50^{th} leftmost digit of N?

3. Six distinct positive integers are randomly chosen between 1 and 2006, inclusive. What is the probability that some pair of these integers has a difference that is a multiple of 5? What if we choose five distinct positive integers?

4. How many integers n satisfy both of the following conditions:

   (a) $100 < n < 200$

   (b) n has the same remainder whether it is divided by 6 or by 8.

5. The sum of the respective remainders of 63, 91, 129 divided by a is equal to 25. Find a.

1.10 Number sense (part 3)

1. Compute the number of integers n for which $2^4 < 8^n < 1632$.

2. Confirm that $(a^2+b^2)(x^2+y^2) = (ax+by)^2 + (ay-bx)^2$. Find another way to write $(a^2+b^2)(x^2+y^2)$ as the sum of two squares.

3. Factor each of the numbers 391, 899, 1591, 2491, 3599, 4209, 5561, and 5609. (A certain algebraic identity is very helpful!)

4. Writing 1313 as the sum of two perfect squares.

   (a) For digits A, B, and C, $\overline{\texttt{AB}}^2 + \overline{\texttt{AC}}^2 = 1313$. Find all possible 3-digit numbers $\overline{\texttt{ABC}}$.

   (a) For digits W, X, Y, and Z, $\overline{\texttt{WX}}^2 + \overline{\texttt{YZ}}^2 = 1313$. Find all possible 4-digit numbers $\overline{\texttt{WXYZ}}$.

   (Consider two approaches. The first is related to approximation and units digit examination. The second is related to a certain algebraic identity.)

5. Eric, Meena, and Cameron are studying the famous equation $E = mc^2$. To memorize this formula, they decide to play a game. Eric and Meena each randomly think of an integer between 1 and 50, inclusively, and substitute their numbers for E and m in the equation. Then, Cameron solves for the absolute value of c. What is the probability that Cameron's result is a rational number?

1.18 Multiples of small numbers (part 1)

1. A particular positive three-digit integer is divisible by 5. The integer also is divisible by 11. The sum of the three digits of the integer is 13. What is the integer?

2. Complete the following sentence:

   If the sum of three integers is divisible by 3, then the remainders of these integers divided by 3 are either ... or ... .

3. Every high school in the city of Euclid sent a team of 3 students to a math contest. Each participant in the contest received a different score. Andrea's score was the median among all students, and hers was the highest score on her team. Andrea's teammates Beth and Carla placed 37^{th} and 64^{th}, respectively. How many schools are in the city?

4. Consider the statement:

   Positive integer $n = \overline{a_h a_{h-1} \ldots a_1 a_0}$ is divisible by 7 if and only if the difference $m = \overline{a_h a_{h-1} \ldots a_1} - 2 \cdot a_0$ is divisible by 7.

   Understand this statement by checking at least 5 numerical examples that no one else in the class will think of. Show that this statement is true.

5. Find the smallest positive integer n such that the decimal representation of $2010n$ has exactly one even digit.