Lectures on Challenging Mathematics

Essential Computational Mathematics
Volume 3.3

PCX Geometry

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“Cogito ergo Sum” – “I think, therefore I am”

René Descartes (1596-1650)

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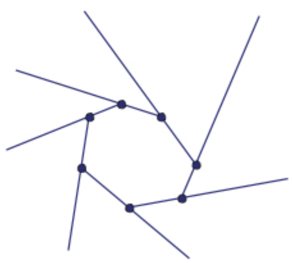
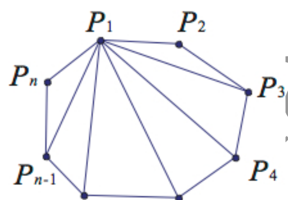
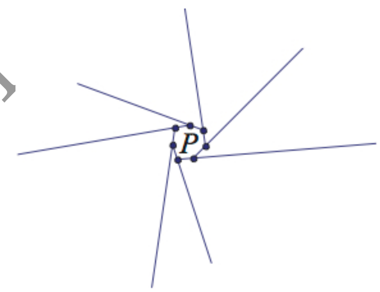
1.5 Sentry theorem (part 1)

1. Alex walks along the boundary of a n-sided plot of land, writing down the number of degrees turned at each corner. What is the sum of these n numbers for $n = 3, 4, 5, \ldots$?

   [Sentry theorem] The sum of the exterior angles (one per vertex) of any polygon is 360 degrees. The sum of interior angles of a n-sided convex polygon is $(n-2) \cdot 180°$.

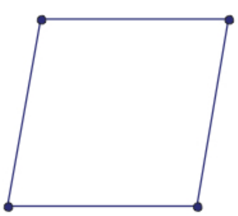


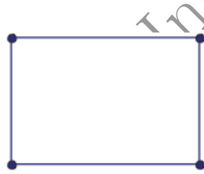


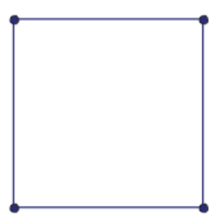


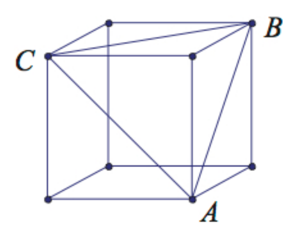
2. The sides of a polygon are cyclically extended to form *rays*, creating one exterior angle at each vertex. Viewed from a great distance, what theorem does this figure illustrate?

3. A polygon is *equilateral* if its sides have the same length. A polygon is *equiangular* if its interior angles are the same size. Clearly, a polygon is equiangular if its exterior angles are the same size. For a triangle, equilateral is equivalent to equiangular. For polygons with more than 3 sides, these two concepts are not equivalent anymore. For example, rhombus is equilateral but not necessarily equiangular. On the other hand, a rectangle is equiangular but not necessarily equilateral. A polygon that is both equilateral and equiangular is called *regular*. Square is the regular quadrilateral.



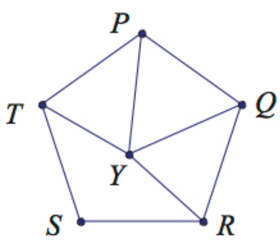


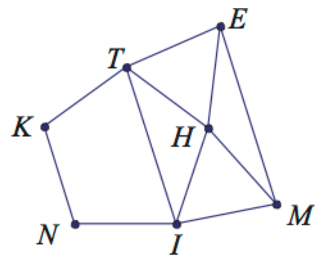




   Find the angle formed by two face diagonals that intersect at a vertex of the cube.

4. Mark Y inside regular pentagon $PQRST$, so that PQY is equilateral. Is RYT straight? Explain.





5. Equilateral triangles THE and HIM are attached to the outside of regular pentagon $THINK$. Is quadrilateral $TIME$ a parallelogram? Justify your answer.

1.7 Why there is no SSA congruence theorem? (part 1)

1. Draw triangles ABC and ADE with vertices having coordinates: $A(0,0)$, $B(2,0)$, $C(5,4)$, $D(4,5)$, $E=(0,8)$. Triangles are clearly not congruent, but they have some congruent parts. List all congruences you can find.

2. Construct triangle ABC such that $AB=3$ and $BC=2$, and $\angle BAC=30^\circ$. Is there only one triangle satisfying the given information? If no, try constructing another one.

3. (Continuation) Solve the previous problem for

   (a) $BC=\dfrac{3}{2}$ (b) $BC=1$ (c) $BC=\dfrac{7}{2}$

4. Explain why there is no SSA congruence theorem. Is there a case, when the SSA congruence can be used to prove that two triangles are congruent?

5. The diagonals of quadrilateral $ABCD$ intersect at M. It is known that $AM=CM$, $AB=CD$, $\angle BAM=10^\circ$ and $\angle DCM=50^\circ$.

   (a) List all congruences between triangles AMB and CMD. Are they congruent?

   (b) Construct point E on DM such that triangles AMB and CME are congruent. Describe triangle CDE.

   (c) Find $\angle AMB$.

1.10 Pythagorean theorem (part 4)

1. An isosceles right triangle is removed from each corner of a square piece of paper to create a rectangle $ABCD$. If $AB = 12$ and $BC = 10$, what is the combined area of the four removed triangles, in square units?

2. Consider triangle ABC with $AB = 6$, $AC = 8$, and $BC = 10$. Denote by AD the altitude from vertex A and by M the midpoint of BC. Find the lengths of AD and DM.

3. In acute triangle ABC, let AD and CF be the altitudes from A and C. It is given that $BC = 25$, $CF = 15$, and $AC = 17$. Find the length of AD.

4. In triangle ABC, $\angle A = 90°$, $\angle B = 75°$, and $BC = \sqrt{3}$. What is the area of triangle ABC?

5. State and prove the converse of the Pythagorean theorem.

1.16 Similarity of triangles (part 1)

1. Two triangles are called *similar* if their vertices can be paired so that

   - Corresponding angles are congruent.
   - Corresponding sides are in proportion. Their lengths have the same ratio.

   When triangles ABC and XYZ are similar, we write $\triangle ABC \sim \triangle XYZ$. This notation implies that $\angle A = \angle X$, $\angle B = \angle Y$, $\angle C = \angle Z$, and

   $$\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX}.$$

   AA Similarity theorem. In two triangles, if two pairs of corresponding angles are congruent, then the triangles are similar.

   Let ABC be a triangle and let P and Q be points on AB such that $AP = 2$, $PQ = 5$, $QB = 2$. The line passing through P and parallel to BC intersects AC in K and the line passing through Q and parallel to AC intersects BC in L. List similar triangles in the diagram and find their ratio of similarity.

2. Let $ABCD$ be a trapezoid, $AB \parallel CD$, with its diagonals intersecting at S. Given that $AB = 18$, $AS = 12$, $BS = 15$, $CS = 20$, find the lengths of DS and CD.

3. Let $ABCD$ be a quadrilateral such that diagonal AC bisects angle $\angle A$, $BC \perp AC$, and $CD \perp AD$. Given that $AD = 9$ and $AB = 25$, find the perimeter of $ABCD$.

4. Let ABC be a triangle and let M and N be the midpoints of AB and AC, respectively. Segments BN and CM intersect at P. Find the ratios $\frac{BP}{PN}$ and $\frac{CP}{PM}$.

5. *SAS Similarity theorem.* If an angle of one triangle is congruent to an angle of another triangle and the sides including those angles are in proportion then the triangles are similar.

   Let $ABCD$ be a quadrilateral with its diagonals intersecting at P such that $\angle BAC = \angle BDC$. You are given that $AB = 10$, $BC = 20$, $AP = CP = 9$, and $BP = 13$.

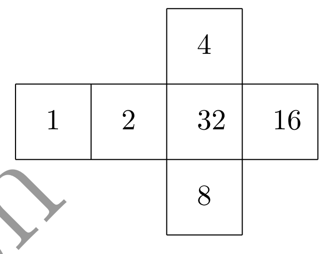
   (a) Find a pair of similar triangles from AA Similarity, then find the length of CD.

   (b) Find another pair of similar triangles from SAS Similarity, then find the length of AD.

2.5 3-D vision (part 2)

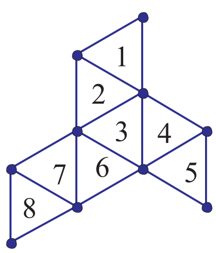
1. Square regions 5 cm by 5 cm are removed from each of the 4 corners of a 30 cm by 40 cm rectangle. The sides are folded up to create an open box. What is the volume, in cubic centimeters, of the interior of the open box?

2. Three cubes are each formed from the pattern in the right-hand side figure. They are then stacked on a table one on top of another so that the 13 visible numbers have the greatest possible sum. What is that sum?



3. How to construct a regular octahedron from a given regular tetrahedron by cutting some of its parts?

4. This net, in the left-hand side figure shown below, is folded into a regular octahedron. What is the sum of the numbers on the triangular faces sharing an edge with the face with 1 written on it?



5. An (regular) *icosidodecahedron*, which has twelve pentagonal faces, is given in the right-hand figure shown above. How many edges does this figure have? How many vertices? How many triangular faces?

2.6 Circles (part 1)

1. Points A, B, C lie on a circle with center P in clockwise order. Given that $\angle APB = 100°$ and $\angle BPC = 140°$, find the measures of angles in triangle ABC.

2. (Continuation) Points D, E, F lie on a circle with center P in clockwise order. Given that $\angle DPE = 34°$ and $\angle EPF = 42°$, find the measures of angles in triangle DEF.

3. A *chord* of a circle is a segment whose endpoints both lie on the circle. A *diameter* is a chord that passes through the center of the circle.

   Let AB be diameter of a circle. If $\angle CAB = 20°$, find $\angle ACB$. What if $\angle CAB = x$?

4. An *inscribed* angle is an angle formed by two chords in a circle which have a common endpoint.

   Points A, B, C, D lie on a circle with center O in clockwise order such that BD is diameter, $\angle ABD = 30°$ and $\angle CBD = 40°$. In this example, $\angle ABD$ subtends arc $\widehat{AD}$ not containing point C, and $\angle AOD$ is the central angle that subtends the same arc. Find $\angle AOD$ and angles of quadrilateral $ABCD$.

5. Points A, B, C, D lie on a circle with center O in clockwise order such that AD is diameter, $\angle BAD = 36°$ and $\angle CAD = 20°$. Find $\angle BOC$ and angles of quadrilateral $ABCD$.

   In view of the problems solved in this section, complete the following observations:

   > Every inscribed angle is equal to _______________ of a central angle that subtends the same arc.
   >
   > Every inscribed angle that subtends a diameter is equal to _____.