Lectures on Challenging Mathematics

Essential Computational Mathematics
Volume 3.2

PCX Counting

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"Cogito ergo Sum" – "I think, therefore I am"

René Descartes (1596-1650)

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1.3 Simple complications of addition and multiplication principles

1. In the counting game Bing-Bong, Arlene starts counting at 1, but skips all multiples of 3 and all numbers that contain the digit 3. For example, Arlene counts: 1, 2, 4, 5, 7, 8, 10, 11, 14, 16 . . . . What is the 40^{th} number in this sequence?

2. Emily is thinking of a positive three-digit integer. All of the digits in her number are prime and distinct. How many possibilities are there for Emily's number? What if we further assume that the digits also increase in order from left to right?

3. Determine the number of positive integers less than 1000 that contain at least one 1 in their decimal representation. (Compare your solution with $999 - 9^3$ and explain.)

4. How many positive three-digit integers with each digit greater than 2 are divisible by 6?

5. Eleven students volunteer for two different community service jobs. Each job needs only two volunteers though (four different people). In how many ways can the jobs be assigned? What if the jobs are the same?

1.4 Dealing with overcounting

1. Helen must read five books for her literature course. She may read any one of three biographies, any two of four mysteries, and any two of five science fiction books on her list. How many different sets of five books can she choose?

2. Find the number of distinguishable permutations of the letter in the word `MISSISSIPPI`.

3. An $11 \times 11 \times 11$ wooden cube is formed by gluing together 11^3 unit cubes. What is the greatest number of unit cubes that can be seen from a single point?

4. Before Rick can open his gym locker, he must recall the combination of three numbers. Two of the numbers are 17 and 24, but he has forgotten the third, and does not know the order of the numbers. There are 40 possibilities for the third number (from 0 to 39). At ten seconds per try, at most how long will it take him to exhaust all possibilities?

5. How many three-digit positive integers have the property that exactly two of the integer's digits are equal? How many of them are even?

1.11 Does the order matter? (part 1)

1. Three distinct vertices are randomly chosen among the vertices of a cube. What is the probability that they are the vertices of an equilateral triangle?

2. A license plate in a certain state consists of three digits, not necessarily distinct, and three letters, also not necessarily distinct. These six characters may appear in any order, except that the three letters must appear next to each other. How many distinct license plates are possible?

3. A license plate in a certain state consists of three digits, not necessarily distinct, and three letters, also not necessarily distinct. These six characters may appear in any order, except that no two letters can be next to each other. How many distinct license plates are possible?

4. A license plate in a certain state consists of three digits, not necessarily distinct, and three letters, also not necessarily distinct. These six characters may appear in any order, except that some letters must be next to each other. How many distinct license plates are possible?

5. A license plate in a certain state consists of three digits, not necessarily distinct, and three letters, also not necessarily distinct. These six characters may appear in any order. How many distinct license plates are possible? How does your answer relate to the answers to previous problems?

1.15 Seeing subtle differences (part 1)

1. A positive integer is said to be *bi-tri-digital* if it uses two different digits, with each digit used exactly three times. How many bi-tri-digital numbers are there?

2. A positive integer is said to be *tri-digital* if it uses three different digits, with each digit used exactly three times. How many tri-digital numbers are there?

3. Seven pebbles are numbered $1, 2, \ldots, 7$. Alex came, chose some (could be zero or all) of the pebbles, painted a red stripe on each of them (below the pebbles number), and then left. Blair than came, chose some (could be zero or all) of the pebbles, painted a blue stripe on each of them (again, below the pebbles number), and then left. How many coloring schemes for these pebbles are possible? (Note that some pebbles can be purple, meaning that a red stripe *and* a blue stripe were painted.)

4. Seven kind of fruits (apple, banana, grapefruit, orange, pear, peach, pineapple) are sold in a store. A banana costs a cent, an orange costs a nickle, an apple costs a quarter, a pear costs \$1, a peach costs \$3, a grapefruit costs \$9, and a pineapple costs \$27.

   Cory spends \$$x$ to buy one or more fruits, all of which are different from one another. Dylan spends \$$y$ to buy one or more fruits, all of which are different from one another. Determine the number of possible pairs of positive real numbers (x, y).

5. (Continuation) Together, Cory and Dylan spend a total of \$$z$ at the store buying fruits, as described in the previous problem. How many different values are possible for z?