Lectures on Challenging Mathematics

Essential Computational Mathematics Volume 3.1

PCX Algebra

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"Cogito ergo Sum" – "I think, therefore I am"

René Descartes (1596-1650)

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1.4 Word problem review (part 2)

1. A traffic light runs repeatedly through the following cycle: green for 30 seconds, then yellow for 3 seconds, and then red for 30 seconds. Leah picks a random three-second time interval to watch the light. What is the probability that the color changes while she is watching?

2. Given that three shirts cost d dollars,

   (a) How many dollars does one shirt cost?

   (b) How many dollars do k shirts cost?

   (c) How many shirts can be bought with q quarters?

3. In a collection of red, blue, and green marbles, there are 25% more red marbles than blue marbles, and there are 60% more green marbles than red marbles. Suppose that there are r red marbles. What is the total number of marbles in the collection?

4. To find a proper solution to the following problem, you might need to make some assumptions, what could be the assumptions.

   > If a hen and a half can lay an egg and a half in a day and a half, then how much time is needed for three hens to lay three eggs?

   You might feel more comfortable with the following (similar) problem instead.

   > If ten (identical) hoses can fill a empty pool of 1000 gallons capacity in 10 hours, then how much time is needed for three such hoses to fill a empty pool of 300 gallons capacity?

5. Gary purchased a large beverage, but drank only $\frac{m}{n}$ of this beverage, where m and n are relatively prime positive integers. If Gary had purchased only half as much and drunk twice as much, he would have wasted only $\frac{2}{9}$ as much beverage. Find $m + n$.

1.6 The difference of the squares (part 1)

1. Confirm that $a^2 - b^2 = (a+b)(a-b)$, expand $\left(\sqrt{s}+\sqrt{t}\right)\left(\sqrt{s}-\sqrt{t}\right)$, and factor $x^4 - y^4$.

2. For every positive integer n, express the sum of even positive integers up to $2n$:

$$2 + 4 + 6 + \cdots + 2n$$

in a compact form.

Then express in a compact form the sum of odd positive integers up to $2n+1$:

$$1 + 3 + 5 + \cdots + (2n+1)$$

in a compact form. (Hint: use difference of squares.)

3. *Rationalize the denominator* (so that there is no reference to the radicals in the denominator) for each of the following expressions:

(a) $\dfrac{1}{2+\sqrt{3}}$ (b) $\dfrac{1}{2-\sqrt{3}}$ (c) $\dfrac{2+\sqrt{2}}{3+\sqrt{5}}$ (d) $\dfrac{\sqrt{32}}{\sqrt{3}+\sqrt{2}}$

4. Evaluate $2003 \cdot 1993 - 1999 \cdot 1997$.

5. For positive integer m and n with $m < n$.

(a) Show that the sum

$$S_{m,n} = (m+1) + (m+2) + \cdots + (n-1) + n = \frac{n(n+1)}{2} - \frac{m(m+1)}{2}.$$

(b) Factor the right-hand side expression in the above equation.

(c) Explain why we can obtain this factored form directly.

(d) Determine if $S_{m,n}$ is always divisible by $n - m$.

1.8 Linear graphs and convex regions (part 1)

1. Victor's tennis club membership costs him \$200 per year and he has to pay an additional \$3 per hour on court rental fees. The town courts cost a flat fee of \$6 per hour to rent.

   (a) Write an equation that expresses Victor's total cost, C, of playing h hours of tennis in a given year at my tennis club.

   (b) Write an equation that expresses Victor's total cost, T, of playing h hours of tennis in a given year at town courts.

   (c) For how many hours of tennis should Victor play so it is worth to keep his membership at his tennis club?

   (d) Draw the graph of *piecewise function*

   $$C = \begin{cases} 6h, & 0 \le h \le \frac{200}{3}, \\ 200 + 3h, & \frac{200}{3} \le h. \end{cases}$$

   How does this graph relate to Victor's total cost of paying tennis?

   (e) Draw a graph of Victor's average cost of playing tennis per hour.

2. Let n denote the number of the non-overlapping regions that can be formed by three lines in the plane. What are the possible values of n?

3. Let $X = (3, 5)$, $Y = (7, -11)$, $Z = (-13, -17)$.

   (a) Write an point-slope form equation for each of the lines XY, YZ, ZX;

   (b) Find a_1 and b_1 such that points $P_1 = (a_1, 2)$ and $P_2 = (-3, b_1)$ lie on line XY.

   (c) Find a_2 and b_2 such that points $Q_1 = (a_2, -4)$ and $Q_2 = (5, b_2)$ lie on line YZ.

   (c) Find a_3 and b_3 such that points $R_1 = (a_3, -5)$ and $R_2 = (-6, b_3)$ lie on line ZX.

4. Lines XY, YZ, ZX cut the coordinate plane into seven regions. Locate each of the points $A = (0, 0)$, $B = (1, 12)$, $C = (-7, -7)$, $D = (-2, -13)$, $E = (11, -10)$, $F = (5, 7)$, $G = (-16, -16)$ into these seven regions.

5. In a coordinate plane, shade the region that consists of all points that have positive x- and y-coordinates whose sum is less than 5. Write a system of three inequalities that describes this region.

1.20 The first look at quadratic functions and parabolas (part 2)

1. Graph the equations on the same system of axes: $y = x^2$, $y = 0.5x^2$, $y = 2x^2$, and $y = -x^2$. What is the effect of a in equations of the form $y = ax^2$?

2. Avery and Sasha were comparing parabola graphs on their calculators. Avery had drawn $y = 0.001x^2$ in the window $-1000 \le x \le 1000$ and $0 \le y \le 1000$, and Sasha had drawn $y = x^2$ in the window $-k \le x \le k$ and $0 \le y \le k$. Except for scale markings on the axes, the graphs looked exactly the same! What was the value of k?

3. You have seen that the graph of any quadratic function is a parabola that is symmetrical with respect to a line called the *axis of symmetry*, and that each such parabola also has a lowest or highest point called the *vertex*. Sketch a graph for each of the following quadratic functions. Identify the coordinates of each vertex and write an equation for each axis of symmetry.

   (a) $y = 2x^2 - 5$ (b)$y = 5 - x^2$ (c) $y = x^2 + 5x$ (d) $y = x^2 - 2x - 3$

4. When asked to solve the equation $(x-3)^2 = 11$, Jess said, "That's easy - just take the square root of both sides." Perhaps Jess also remembered that 11 has two square roots, one positive and the other negative. What are the two values for x, in exact form?

   However, when asked to solve the equation $x^2 - 8x = 1$, Jess said, "Hmm ... not so easy, but I think that adding something to both sides of the equation is the thing to do." This is indeed a good idea, but what number should Jess add to both sides? How is this equation related to the previous one?

5. Using a driver on the 8th tee, which is on a plateau 10 meters above the level fairway, Dale hits another fine shot. Explain why the quadratic function $y = 10 + 0.5x - 0.002x^2$ describes this parabolic trajectory. Why should you expect this tee shot to go more than 250 meters? To find the length of the shot, follow these steps:

   (a) Set y equal to 0 and solve for x. Explain why, and show how to arrive at $x^2 - 250x = 5000$.

   (b) Add 125^2 to both sides of this equation. Why was this number chosen?

   (c) Complete the solution and find the length of the shot.

   (d) Comment on the presence of the number 125 in the answer. What is its significance?

1.25 Algebra practice set 3

1. Solve $4^{2013} - 4^{2012} - 4^{2011} + 4^{2010} = 45\left(2^x\right)$ for x.

2. The base of a rectangular tank is five feet by six feet, and the tank is three feet tall. The water in the tank is currently nine inches deep.

   (a) How much water is in the tank?

   (b) The water level will rise when a one-foot metal cube (denser than water) is placed on the bottom of the tank. By how much?

   (c) The water level will rise some more when a second one-foot metal cube is placed on the bottom of the tank, next to the first one. By how much?

3. (Continuation) Several one-foot metal cubes (denser than water) are placed on the bottom of the tank so that all the cubes are barely under the water. How deep is the water?

4. If x, y, and z are positive real numbers satisfying

$$x + \frac{1}{y} = 4, \quad y + \frac{1}{z} = 1, \quad \text{and} \quad z + \frac{1}{x} = \frac{7}{3},$$

   then what is xyz?

5. Let $\mathcal{C}$ denote the graph of $5y = |12x - 4|$. The graph $\mathcal{C}_1$ is obtained by moving $\mathcal{C}$ down vertically by 1 and then reflecting across the line $y = 3$. The graph $\mathcal{C}_1$ is obtained by reflecting across the line $y = 3$ and then moving $\mathcal{C}$ down vertically by 1.

   (a) Find an equation for the graph $\mathcal{C}_1$ and compute the perimeter of the region enclosed by $\mathcal{C}_1$ and $\mathcal{C}$.

   (b) Find an equation for the graph $\mathcal{C}_2$ and compute the area of the region enclosed by $\mathcal{C}_2$ and $\mathcal{C}$.