1.6 Angle chasing and centers of triangles – computations and proofs (part 3)

1. Let $ABCD$ be a convex quadrilateral with $AB < AD$. Diagonal AC bisects $\angle BAD$, and $\angle ABD = 130^\circ$ and $\angle BAD = 40^\circ$. Let E be a point on the interior of segment AD. Given that $BC = CD = DE$, determine $\angle ACE$ in degrees.

2. [Via Kevin Sun] Triangle ABC is inscribed in circle ω. Let M be the midpoint of arc $\widehat{AB}$ not including C. Show that $CM^2 = CA \cdot CB + MA \cdot MB$.

3. [New Problems in Euclidean Geometry, by David Monk] Let $ABCD$ be a cyclic quadrilateral, and let E be the midpoint of side BC. Point X and Y lie on lines AB and CD, respectively, such that $XE \perp BC$ and $EY \perp AD$. Prove that $XY \perp CD$.

4. Triangle ABC, with incenter I, is inscribed in circle ω. Rays AI, BI, CI meet ω again (other than A, B, C) at D, E, F, respectively. Let ℓ_F, ℓ_D, ℓ_E denote the tangent lines to ω at F, D, E, respectively. Lines ℓ_F and AI, ℓ_D and BI, ℓ_E and CI meet in P, Q, R. Prove that $AP \cdot BQ \cdot CR = ID \cdot IE \cdot IF$.

5. In triangle ABC, points P and Q lie on sides AB and AC respectively such that $BP = CQ$. Let M and N be the respective midpoints of segments BC and PQ. Prove that line MN is parallel to the bisector of $\angle A$.

1.17 Euler's formula

1. Let ABC be a triangle with circumradius R and inradius r. Let O and I denote the circumcenter and incenter of the triangle, respectively. Triangle ABC is inscribed in circle ω. Ray AI meets ω again at M (other than A). Set $d = OI$. Express $AI \cdot IM$ in terms of d and R.

2. (Continuation) Set $\angle A = \angle BAC$. Express AI and IM in terms of R, r, $\angle A$.

3. (Continuation) Establish the result in the previous problem with a synthetic approach. (One might want to consider diameter MN of ω.)

4. (Continuation) Find a relation between R, r, d. (This is the simplest case of *Poncelet's porism*, and is sometimes also known as *Euler's triangle theorem*.) Deduce that $R \geq 2r$.

5. Consider two distinct triangles T_1 and T_2 have the same circumcircle. Determine if it is possible that the incircle of T_1 lies inside the incircle of T_2.

1.18 Selected entry level Olympiad geometry problems (part 2)

1. In an acute triangle ABC, point H is the orthocenter and A_0, B_0, C_0 are the midpoints of the sides BC, CA, AB, respectively. Consider three circles passing through H: ω_a around A_0, ω_b around B_0 and ω_c around C_0. The circle ω_a intersects the line BC at A_1 and A_2; ω_b intersects CA at B_1 and B_2; ω_c intersects AB at C_1 and C_2. Show that the points $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a circle.

2. Let I denote the incenter of the triangle ABC. A point P lies in the interior of triangle ABC satisfying the equation.

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

   Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

3. Let ABC be a triangle with incenter I, and let D, E, F be the midpoints of sides BC, CA, AB, respectively. Lines BI and DE meet at P, and lines CI and DF meet at Q. Line PQ meets sides AB and AC at T and S, respectively. Prove that $AS = AT$.

4. Let ABC be an acute-angled triangle, and let P and Q be two points on side BC. Construct point C_1 in such a way that convex quadrilateral $APBC_1$ is cyclic, $QC_1 \parallel CA$, and C_1 and Q lie on opposite sides of line AB. Construct point B_1 in such a way that convex quadrilateral $APCB_1$ is cyclic, $QB_1 \parallel BA$, and B_1 and Q lie on opposite sides of line AC. Prove that points B_1, C_1, P, and Q lie on a circle.

5. Points P and Q lie on side BC of acute triangle ABC so that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on lines AP and AQ, respectively, such that $AP = PM$ and $AQ = QN$. Prove that lines BM and CN intersect on the circumcircle of triangle ABC.

1.22 Tiles and coloring (part 4)

1. Let a and b be given positive real numbers. The equation $y - b = m(x - a)$ represents a line ℓ_m for any real number m. For $m < 0$, consider the triangle OX_mY_m where $O = (0, 0)$ and X_m and Y_m are the x-intercept and y-intercept of ℓ_m, respectively.

   (a) Express $[OX_mY_m]$, the area of triangle OX_mY_m, in terms of a, b, and m.

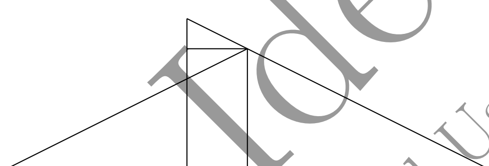
   (b) Show algebraically that $[OX_mY_m] \geq 2ab$, where the equality holds when $m = m_0 = -\frac{b}{a}$.

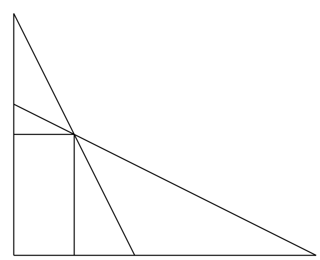
   (c) Note that $m = m_0$ if and only if $P = (a, b)$ is the midpoint of the segment X_mY_m. We can also show that $[OX_mY_m] \geq 2ab$ geometrically. Draw lines ℓ_{m_0} and ℓ_{m_1}, where m_1 is some negative number. Show that $[OX_{m_1}Y_{m_1}] \geq [OX_{m_0}Y_{m_0}]$.

2. (Continuation) We can summarize the fact we have just established in the previous problem into a mathematical statement. One such statement can start with "A *given* rectangle is inscribed in a right triangle such that one of the corners of the rectangle is at the vertex of the right triangle with the right angle, ..." Complete this statement.

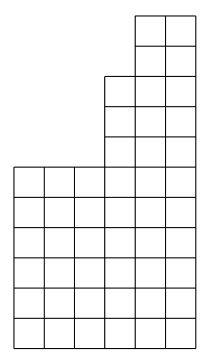
   What is the *dual statement* of the above statement? That is, the statement starts with "A rectangle is inscribed in a *given* right triangle such that one of the corners of the rectangle is at the vertex of the right triangle with the right angle, ..." Complete this statement.

3. The diagrams shown below provide some hints to establish the result in the previous problem. This kind of method is often referred as *proof without words*.





4. The polygon shown in the right-hand side can be transformed into a square by dissection. There are (at least) two ways to achieve this by cutting the polygon into three pieces. Find these two ways.



5. Given any finite collection of squares of total area at least 4, show that one can cover a unit square with the given squares.