2.9 Math reasoning practice set 3

1. For a positive integer n, let $C(n)$ denote the number of pairs of consecutive 1s in the binary representation of n. (For example, $C(183) = C(10110111_2) = 3$). Compute $C(1) + C(2) + C(3) + \cdots + C(256)$.

2. Consider the sequence
$$a_n = \left(1 + \frac{1}{n}\right)^n \quad \text{for } n = 1, 2, \ldots.$$
Show that this sequence is
   (a) bounded, that is, there is a constant M such that $a_n \le M$ for every positive integer n.
   (b) a strictly increasing; that is, to show that $a_n < a_{n+1}$ for every positive integer n. (This result can be established with or without AM-GM inequality. Try both ways.)

   Parts (a) and (b) lead to the existence of the number e, one the most important constant in mathematics.)

3. For positive integer n, let $(a_1, a_2, \ldots, a_n)$ be a permutation of $(1, 2, \ldots, n)$. Consider
$$f(a_1, a_2, \ldots, a_n) = ||\cdots||a_1 - a_2| - a_3| - \cdots| - a_n|$$
   (a) Show that the parity of $f(a_1, a_2, \ldots, a_n)$ depends only on n (rather than the permutation $(a_1, a_2, \ldots, a_n)$.
   (b) Show that $f(a_1, a_2, \ldots, a_n) \le n$.
   (c) For each of $n = 2010, 2011, 2012, 2013$, determine the minimum and maximum values of $f(a_1, a_2, \ldots, a_n)$.

4. Let $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ be a set of vectors with the sum of their magnitudes equal to 1. Determine if the following statement is true:

   > There is a subset $\mathcal{W}$ of $\mathcal{V}$ such that magnitude of the sum of the vectors in $\mathcal{W}$ is at least $\frac{1}{6}$.

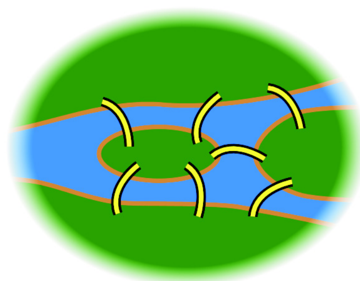
   (Hint: Dissect the plane into three congruent regions.)

5. There are sixteen 100-gram balls and sixteen 99-gram balls on a table (the balls are visibly indistinguishable). You are given a balance scale with two sides that reports which side is heavier or that the two sides have equal weights. A weighing is defined as reading the result of the balance scale: For example, if you place three balls on each side, look at the result, then add two more balls to each side, and look at the result again, then two weighings have been performed. You wish to pick out two different sets of balls (from the 32 balls) with equal numbers of balls in them but different total weights. What is the minimal number of weighings needed to ensure this?

2.11 Eulerian walks and Hamiltonian cycles

1. The Seven Bridges of Königsberg problem is a historically notable problem in mathematics. Its negative resolution by Leonhard Euler in 1735 laid the foundations of graph theory and prefigured the idea of topology.

   The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands which were connected to each other and the mainland by seven bridges. (Please see the figure shown below. This figure and the following introduction of the problem as edited based on the information given at wikipedia.)

   The problem was to find a walk through the city that would cross each bridge once and only once. (The islands could not be reached by any route other than the bridges, and every bridge must have been crossed completely every time; one could not walk halfway onto the bridge and then turn around and later cross the other half from the other side. The walk need not start and end at the same spot.) Euler proved that the problem has no solution. There could be no non-retracing the bridges. The difficulty was the development of a technique of analysis and of subsequent tests that established this assertion with mathematical rigor.

   Reformulate the above problem by graph theory terminologies.

2. In graph theory, an *Eulerian walk* (or *Eulerian trail*) is a walk in a graph which visits every edge exactly once. Similarly, an *Eulerian cycle* (or *Eulerian circuit*) is an Eulerian walk which starts and ends on the same vertex. They were first discussed by Leonhard Euler while solving the famous Seven Bridges of Königsberg problem in 1736. Euler proved that a necessary condition for the existence of Eulerian circuits is that all vertices in the graph have an even degree. (Hence the answer of the Königsberg problem is negative.) Now it is your turn to prove this result. More precisely, show that if a connected graph has more than two nodes with odd degree, then it has no Eulerian walk.

   Euler stated without proof that this condition is also sufficient; that is, a connected graph with all vertices of even degree has an Eulerian circuit. The first complete proof of this latter claim was published posthumously in 1873 by Carl Hierholzer. The term Eulerian graph has two common meanings in graph theory. One meaning is a graph with an Eulerian circuit, and the other is a graph with every vertex of even degree. These definitions are equivalent to each other for connected graphs. In the next few problems, we establish this equivalence relation.

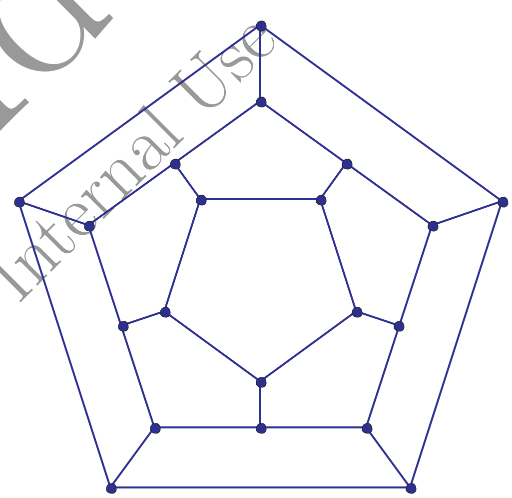
3. Let G be a connected graph, and let v be a fixed vertex in G. Consider the set of all closed walk starting and ending at v that uses every edge at most once. Suppose that this set is nonempty, and let W be such a walk in this set with the maximum number of edges. Further assume that there is an edge not belonging to W. Prove that there is an edge not belonging to W that is adjacent to a vertex in W.

4. Show that there is an Eulerian walk in a given connected graph with no odd-degree vertices, and every Eulerian walk in this graph is a Eulerian cycle.

5. A *Hamiltonian path* is a path in an undirected graph that visits each vertex exactly once. A *Hamiltonian cycle* (or *Hamiltonian circuit*) is a Hamiltonian path that is a cycle.

   A *Hamiltonian path* or *traceable path* is a path that visits each vertex exactly once. A graph that contains a Hamiltonian path is called a *traceable graph*. A *Hamiltonian cycle*, or *Hamiltonian circuit*, or *vertex tour* is a cycle that visits each vertex exactly once (except the vertex that is both the start and end, and so is visited twice). A graph that contains a Hamiltonian cycle is called a *Hamiltonian graph*. (There is no particular known methods to determining whether a graph is Hamiltonian. Determining whether Hamiltonian paths and cycles exist in graphs is the *Hamiltonian path problem*, which is NP-complete.)

   Hamiltonian paths and cycles and cycle paths are named after William Rowan Hamilton who invented the Icosian game, now also known as Hamilton's puzzle, which involves finding a Hamiltonian cycle in the edge graph of the dodecahedron. (See the figure shown below.) Now it is your turn to find it. (Before you do that, are you convinced that this is the planar graph of a dodecahedron?)



2.22 Mathematical arguments – the well ordering principle

1. Let S be a set of finitely many points on a plane such that, for any pair of points A and B in S, there is a third point C in S to form a equilateral triangle ABC. Determine with proof the maximum number of elements in S.

2. Let $x_1, x_2, \ldots, x_n$ be real numbers with $x_1 + x_2 + \cdots + x_n = 0$. Prove that there is an index i such that $x_i + x_{i+1} + \cdots + x_{i+j} \geq 0$ for $j = 0, 1, 2, \ldots, n$, where $x_{n+k} = x_k$.

3. Let n be an integer greater than 1. Suppose $2n$ points are given in the plane. Suppose n of the given $2n$ points are colored blue and the other n colored red. A segment in the plane is called *balanced* if it connects a red point and a blue point. Prove that one can construct n nonintersecting balanced segments.

4. Given regular polygon $A_1A_2 \cdots A_n$ inscribed in circle ω and point P inside the circle, prove that there exist vertices A_i and A_j such that $\angle A_iPA_j \geq (1 - \frac{1}{n}) \cdot 180^\circ$.

5. In an arena, each row has 199 seats. One day, 1990 students are coming to attend a soccer match. It is only known that at most 39 students are from the same school. If students from the same school must sit in the same row, determine the minimum number of rows that must be reserved for these students.

   Interested readers might want to solve this dual version of the problem:

   > In an arena, there are 11 rows of seats and each row has 199 seats. One day, n students are coming to attend a basketball match. It is only known that at most 39 students are from the same school. If students from the same school must sit in the same row, determine the maximum number of students such that all the students will be seated.