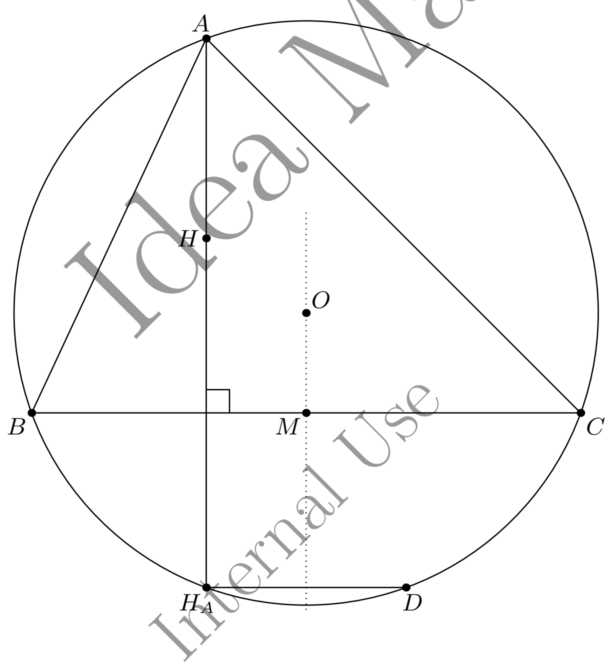
1.9 Angle chasing and the centers of triangles (part 2)

1. The altitudes BE and CF in triangle ABC intersect in point H. Denote by O the circumcenter of triangle ABC. Prove that line AO is perpendicular to line EF.

2. The altitudes AA_1, BB_1, CC_1 of an acute triangle ABC intersect in point H. Show that the orthocenter of triangle ABC, H, is the incenter of triangle $A_1B_1C_1$.

3. (Continuation) Suppose ABC is an obtuse triangle with $\angle A > 90°$. Describe the orthocenter of triangle ABC in relation with triangle $A_1B_1C_1$.

4. Let ABC be an acute triangle with circumcircle ω. Let O and H denote its circumcenter and orthocenter. Denote by M be the midpoint of BC. Point H_A lies on $\widehat{BC}$ (not including A) such that $AH_A \perp BC$. Let D be the reflection of H_A across line OM.



   Prove that

   (a) H and H_A are symmetric across the line BC.

   (b) points H, M, and D are collinear.

5. In parallelogram $ABCD$, point H is the orthocenter of triangle ABC. The line through H parallel to line AB meets line BC at P and AD at Q; the line through H parallel to line BC meets line AB at R and CD at S. Prove that P, Q, R, S lie on a circle.

1.20 Challenges in geometry computations (part 8)

1. Triangle ABC has $AB = 2AC$. Let D and E be points on segments AB and BC, respectively, such that $\angle BAE = \angle ACD$. Let F be the intersection of segments AE and CD, and suppose that triangle CFE is equilateral. What is $\angle ACB$?

2. In triangle ABC, point D lies on side AC so that $\angle ABD = \angle CBD$. Point E lies on line AB and lines CE and BD intersect at P. Given that quadrilateral $BCDE$ is cyclic, $BP = 12$ and $PE = 4$, compute the ratio $\dfrac{AC}{AE}$.

   One of the conditions of the problem is unnecessary. Which one?

3. A bubble in the shape of a hemisphere of radius 1 is on a tabletop. Inside the bubble are five congruent spherical marbles, four of which are sitting on the table and one which rests atop the others. All marbles are tangent to the bubble, and their centers can be connected to form a pyramid with volume V and with a square base. Compute V.

4. Let T_1 be a triangle with sides $2011, 2012$, and 2013. For $n \geq 1$, if $T_n = \triangle ABC$ and D, E, and F are the points of tangency of the incircle of triangle ABC to the sides AB, BC, and AC, respectively, then T_{n+1} is a triangle with side lengths AD, BE, and CF, if it exists. What is the perimeter of the last triangle in the sequence $\{T_n\}$?

5. In triangle ABC, $\angle BAC = 120°$. The angles bisectors of angles A, B, and C meet the opposite sides at D, E, and F, respectively. Compute $\angle EDF$.

2.4 Tessellations and tiles (part 2)

1. Is it possible for a pentagon to have interior angles 120°, 120°, 120°, 90°, and 90°, *in this order*? Do any of such pentagons tessellate (the plane)? (Note that the pentagon does not need to be equilateral.)

   What about 120°, 120°, 90°, 120°, and 90°? Do any of such pentagons tessellate (the plane)? (Note that the pentagon does not need to be equilateral.)

   By the way, are there other arrangements of the five angles that could have been considered?

2. Find

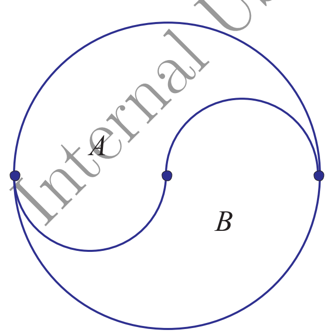
   (a) a convex quadrilateral that can be tiled by 100 congruent triangles.

   (b) two triangles that are not similar to each other and each triangle can be tiled by 3 congruent triangles.

3. Consider the heptagon with vertices $(0,0), (50,0), (50,40), (35,30), (35,80), (15,80), (0,70)$. Ravis wants to cut the polygon into pieces that can be reassembled into a rectangle. Determine the minimum number of pieces needed for Ravi to achieve his goal.

4. One can dissect a 5×5 chessboard into a few pieces such that these pieces can be reassembled to form a 3×3 chessboard and a 4×4 chessboard (so the fields of the chessboard are preserved) via *translations*. Achieve this task with as few pieces as possible.

5. A pseudo *Yinyang* symbol is shown below: In a circle, two semicircles, each with half the radius of that of the circle, cut the interior of the circle into two congruent regions A and B. Show that one can draw is line to bisect the areas of both regions A and B simultaneously.



2.5 Compass and straightedge constructions (part 2)

1. A *ruler* is a straightedge that has markings defining a unit length segment. That is using a ruler we can draw a segment of length 1.

   Given segments of lengths a and b, construct with the a ruler and a compass a segment of length ab.

2. Let A, X, B be points lying on a line p (in this order) such that $AX = a$, $BX = b$. Construct a line q that passes through X and is perpendicular to p. Find the midpoint of AB, M. Using compass we can construct point C on q such that A, B, and C are equidistant from M. Find the distance CX.

3. Given segments of length a and b, use compass and a straightedge to construct a segment of length $\sqrt{a^2 + ab + b^2}$.

4. Use a compass and a straightedge to construct the following regular pentagon and a regular dodecagon.

5. Given two segments of lengths a and b, construct with an unmarked straight edge and a compass a segment of length $\sqrt[4]{a^4 + b^4}$.