Lectures on Challenging Mathematics

Algebra 1

Part 2

Summer 2017

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1.5 Linear systems and word problems

1. Farmer MacGregor wants to know how many cows and ducks are in the meadow. He counted 56 legs and 17 heads. How many cows and ducks are there?

2. At the Exeter Candy Shop, Jess bought 5.5 pounds of candy — a mixture of candy priced at \$4 per pound and candy priced at \$3.50 per pound. Given that the bill came to \$20.75, figure out how many pounds of each type of candy Jess bought.

3. A large family went to a restaurant for a buffet dinner. The price of the dinner was \$12 for adults and \$8 for children. If the total bill for a group of 13 persons came to \$136, how many children were in the group?

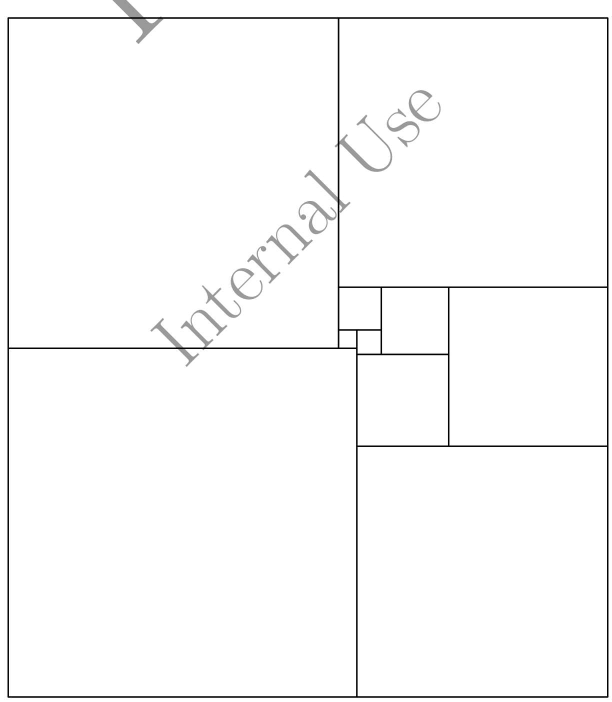
4. Write and graph an equation that states

   (a) that the perimeter of an $l \times w$ rectangle is 768 cm;

   (b) that the width of an $l \times w$ rectangle is half its length.

   Explain how the two graphs show that there is a unique rectangle whose perimeter is 768 cm, and whose length is twice its width. Find the dimensions of this rectangle.

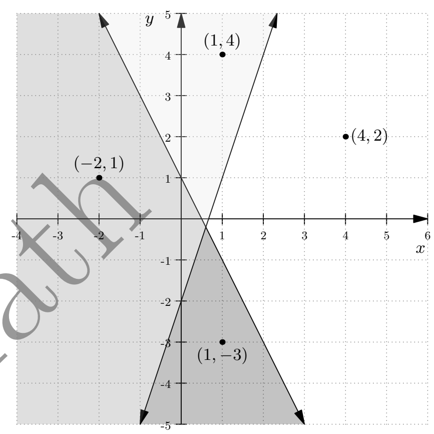
5. The diagram shows a rectangle that has been divided into ten squares of different sizes. The smallest square is 3×3. What are the dimensions of the other squares?



1.9 Systems of linear inequalities

1. Two or more linear inequalities form a *system of linear inequalities* or simply a *system of inequalities*.

   A solution of a system of linear inequalities is an ordered pair that is a solution of each inequality in the system. The *graph* of a system of linear inequalities is the graph of *all solutions* of the system.
   The coordinate plane shows the four regions determined by the lines $3x - y = 2$ and $2x + y = 1$.
   Use the labeled points to help you match each region with one of the systems of inequalities.



   (a) $\begin{cases} 3x - y \le 2 \\ 2x + y \le 1 \end{cases}$ (b) $\begin{cases} 3x - y \ge 2 \\ 2x + y \ge 1 \end{cases}$

   (c) $\begin{cases} 3x - y \ge 2 \\ 2x + y \le 1 \end{cases}$ (d) $\begin{cases} 3x - y \le 2 \\ 2x + y \ge 1 \end{cases}$

2. In a coordinate plane, shade the region that consists of all points that have positive x- and y-coordinates whose sum is less than 5. Write a system of three inequalities that describes this region.

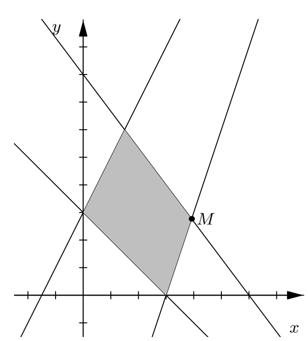
3. Golf balls cost \$0.90 each at Jerzys Club, which has an annual \$25 membership fee. At Rick & Toms sporting-goods store, the price is \$1.35 per ball for the same brand. Where you buy your golf balls depends on how many you wish to buy. Explain, and illustrate your reasoning by drawing a graph.

4. Explain why the system of inequalities $2x - y > 4$ and $2y - 1 > 4x$ has no solutions.

5. The diagram at right shows the graphs of four lines, whose equations are $y = 2x+3$, $x+y = 3$, $4x+3y = 24$, and $3x - y = 9$.

   (a) Find coordinates for the intersection point M.

   (b) Write a system of simultaneous inequalities that describes the shaded region.



Chapter 2

Polynomials and Variables

2.1 Addition and subtraction of polynomials (part 1)

1. A *variable* is a symbol, usually a letter, that is used to represent one or more numbers. In mathematics writing, when a few variables, or a few variables and a number, are written adjacent to each other with no operation symbols between them, then it is implied that they are meant to be multiplied together. A *monomial* is a constant or a product of a constant and variables. If there is only one variable in the monomial, it is a monomial of one variable. Thus 12 is a constant monomial, x^5 is monomial of one variable, $3ax^2$ is a monomial of two variables, but $3xy^4+3x^4y$ is not a monomial. If some variable factors occur more than once, it is customary to use positive integer exponents to *consolidate* them. For example, $3xy = 3\cdot x\cdot y$ and $10xy^2 = 10 \cdot x \cdot y \cdot y$. The *degree* of a monomial counts how many variable factors would appear if it were written without using exponents. For example, the degree of $6ab$ is 2, and the degree of $25x^3$ is 3, because $25x^3 = 25xxx$.

   Rewrite each of these monomials and find their degrees.

   (a) *apple* (b) *banana* (c) *apple* $\cdot$ *banana* (d) *applebanana*
   (e) *bananabread* (f) $x \cdot x^2 \cdot x^3 \cdot x^4$ (g) $(2x)^7$ (h) $2 \cdot x \cdot 3x \cdot (4x)$
   (i) $2x \cdot (3x^2)$ (j) $2 \cdot x \cdot (3x)^2$ (k) $(2w)^3 \cdot 5w^3$ (l) $3a^4 \cdot (\frac{1}{2}b)^3 \cdot ab^6$

2. A *polynomial* in (one variable) is obtained by adding (or subtracting) monomials in the same variable. While $5x$ and $5xy^2$ are both monomials, it is hard to say if $2x + 3x$ or $2xy^2 + 3xy^2$ is a monomial or not, because it is a sum of two *like monomials*. A *binomial* is the sum of two unlike monomials, and a *trinomial* is the sum of three unlike monomials. The monomials that make up a polynomial are often called its *terms*. It is customary to arrange terms in the polynomial in descending degrees from left to right. The constant multiplier in each term is the *coefficient* of the term. The *leading coefficient* of the polynomial is the coefficient of the monomial of the highest degree. For example, we write $x^4 - 2x^5 - 8$ as $-2x^5 + x^4 - 8$, the coefficient of x^4 is 1, and the leading coefficient of the polynomial is -2.

   The sum of two polynomials can be found by writing plus sign between the polynomials and then collect like terms. For example, we can add the binomial $2x^2 - 5$ with trinomial

$2x^3 - x + x^2$ either in *horizontal* format

$$(2x^2 - 5) + (-2x^3 + x - x^2) = -2x^3 + (2x^2 - x^2) + x - 5 = -2x^3 + x^2 + x - 5$$

or in *vertical* format

$$\begin{array}{rrrrr} & & +2x^2 & & -5 \\ +) & -2x^3 & -x^2 & +x & \\ \hline & -2x^3 & +x^2 & +x & -5 \end{array}$$

Find the sum of the following pairs of polynomials. Make sure to apply both vertical and horizontal method for each pair.

(a) $-3x^3 + 2x - 4$, $4x^3 + 2 + 3x^2$

(b) $4ax^2 + a^2x - 5 + 4bx$, $8 - 5bx + 3ax^2 + 2a^2x$

3. The sum of polynomial and its *additive inverse* add to 0. For example, the additive inverse of $2x$ is $-2x$, and the additive inverse of $x^2 - 2$ is $-x^2 + 2$. Find the additive inverse of each of the following polynomials.

(a) $7x^3 - 6x^2 - 4x + 3$

(b) $6a^5 - 5a^4 + 4a^3 - 3a^2 + 2a - 1$

(c) $-6p^4 + 3p^2 - 5$

(d) $2m^7 - (3m^5 - 5m^3 + 7m^2)$

Comment on the following statement:

> The additive inverse of a polynomial can be obtained by replacing each term by its additive inverse.

4. For two polynomials $p(x)$ and $q(x)$, the *difference* $p(x) - q(x)$ is obtained by *subtracting* polynomial $q(x)$ from $p(x)$; that is, adding the additive inverse of $q(x)$ to $p(x)$. For example, for $p(x) = 2x - 3$ and $q(x) = 4x^2 - x + 5$, we have

$$p(x) - q(x) = (2x - 3) - (4x^2 - x + 5) = (2x - 3) + (-4x^2 + x - 5) = -4x^2 + 3x - 8.$$

Find $p(x) - q(x) + r(x)$ for

(a) $p(x) = -9x^5 - x^3 + 2x^2 + 5$, $q(x) = 2x^5 - x^4 + 4x^3 - 3x^2$, $r(x) = 2x^2 - 3x + 5$

(b) $p(x) = 6x^3 - x + 3x^2 - 9$, $q(x) = 2x^2 + 5 - 4x$, $r(x) = 8 - 7x^2 + x^3$

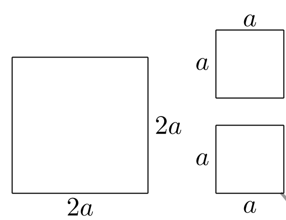
(c) $p(x) = -3m^3 - 4m^2 + 3$, $q(x) = -m^3 + 3m^2 - m - 4$, $r(x) = 4 - m^3 + 2m^2$

(d) $p(x) = \frac{1}{3}x^5 - \frac{x^3}{5}$, $q(x) = \frac{1}{2}x^2 - \frac{2}{3}x + 8$, $r(x) = \frac{x^5}{6} + \frac{1}{3}x^2 - \frac{x^3}{10} - 11$

5. A polynomial is called *prime* if the absolute value of each of its coefficients is a prime number. Find a pair of prime polynomials whose difference is equal to $9y^4 - 3y^3 + 8y^2 - 5y + 2$.

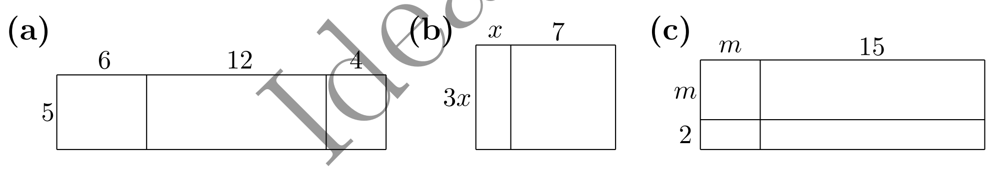
2.7 Areas and polynomial expressions

1. Write $(2x)^2$ without parentheses. Is $(2x)^2$ larger than, smaller than, or the same as $2x^2$? Make reference to the diagram at right in writing your answer. Draw a similar diagram to illustrate the non-equivalence of $(3x)^2$ and $3x^2$.

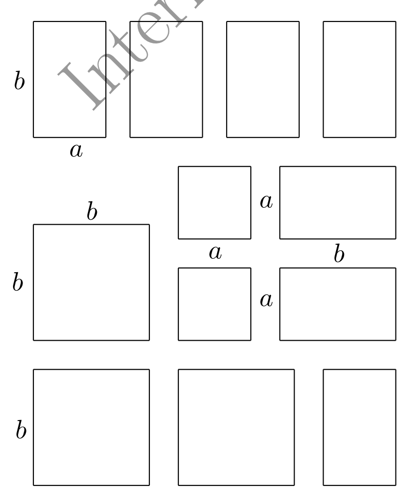


2. If s stands for a perfect square, what formula stands for the next perfect square?

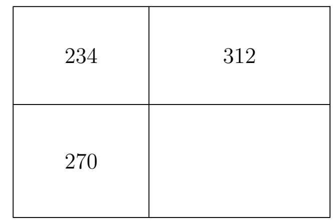
3. Express the areas of the following large rectangles in two ways. First, find the area of each small rectangle and add the expressions. Second, multiply the total length by the total width.



4. All the dimensions of the twelve rectangles in the figure are either x or y. Write an expression for the sum of the areas of the twelve pieces. This should help you to show how these twelve pieces can be fit together to form one large rectangle.



5. The rectangle shown at right has been broken into four smaller rectangles. The areas of three of the smaller rectangles are shown in the diagram. Find the area of the fourth one.



3.6 Envisioning absolute value via word problems (part 2)

1. Rearrange the eight words "between", "4", "the", "17", "is", "and", "x", and "distance" to form a sentence that is equivalent to the equation $|x - 17| = 4$. By working with a number line, find the values of x that fit the equation.

2. Asked to solve the inequality $3 < |x - 5|$ at the board, Corey wrote "$8 < x < 2$," Sasha wrote "$x < 2$ or $8 < x$," and Avery wrote "$x < 2$ and $8 < x$." What do you think of these answers? Do any of them agree with your answer?

3. Graph $y = |x| + 3$ and $y = |x| - 5$, then describe in general terms how the graph of $y = |x|$ is transformed to produce the graph of $y = |x| + k$. How can you tell from the graph whether k is positive or negative?

4. Write a mathematical expression using absolute value that says

   (a) "x is at least 4 units but no more than 5 units from -3."

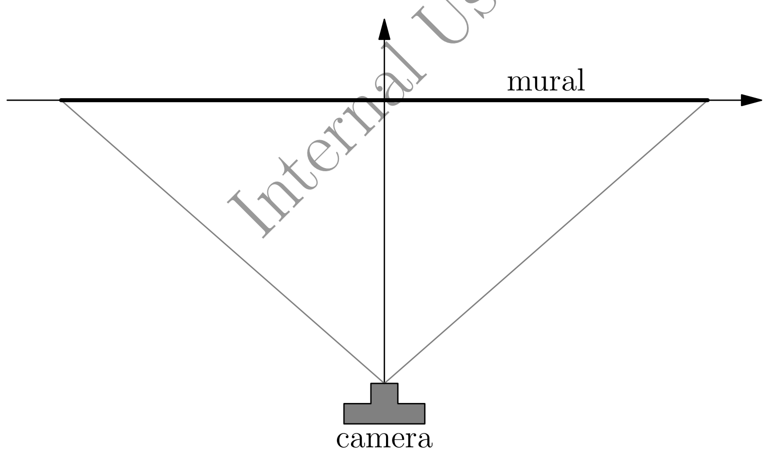
   (b) "x is more than 6 units but less than 8 units from 4."

   (c) "x is at least 7 units but less than 10 units from -5."

   (d) "x is more than 9 units but no more than 13 units from -5."

   Solve each expression and graph your solutions on a number line.

5. Using the coordinate-axis system shown in the top view at right, the viewing area of a camera aimed at a mural placed on the x-axis is bordered by $y = \frac{7}{8}|x| - 42$. The dimensions are in feet. How far is the camera from the x-axis, and how wide a mural can be photographed?



3.7 Factorization of quadratic polynomials

1. The task of the section is to learn how to factor given quadratic polynomial; that is, express a quadratic polynomial as a product of nonconstant linear polynomials with integer coefficients. Let's start with confirming the expansion:

$$(x+p)(x+q) = x^2 + (p+q)x + pq.$$

If quadratic polynomial $x^2 + bx + c$ can be factored into the product $(x+p)(x+q)$ (that is, $x^2 + bx + c = (x+p)(x+q)$), then what are b and c, in terms of p and q?

2. In this problem, we deal with the factorization of quadratic polynomials in general form. For example, we want to find polynomials $(ax+b)(cx+d) = 6x^2 + 17x + 12$. We have, by expanding, the following form:

$$\begin{array}{rr} & ax \;+\; b \\ \times) & cx \;+\; d \\ \hline & bd \\ & (ad)x \\ & (bc)x \\ +) & (ac)x^2 \\ \hline & 6x^2 + 17x + 12 \end{array}$$

Because $ac = 6$, the possible choices for $\{a, c\}$ are $\{\pm 1, \pm 6\}$ and $\{\pm 2, \pm 3\}$. Because $bd = 12$, the possible choices for $\{b, d\}$ are $\{\pm 1, \pm 12\}$, $\{\pm 2, \pm 6\}$, $\{\pm 3, \pm 4\}$. By checking with the condition $ad + bc = 17$, it is not difficult to find that $(a, b, c, d) = (2, 3, 3, 4)$.

Repeat the above the process to factor $2x^2 - 3x - 20$.

3. For each of the following polynomials, determine if it can be factored. If *yes*, factor it; if *no*, state that they cannot be factored.

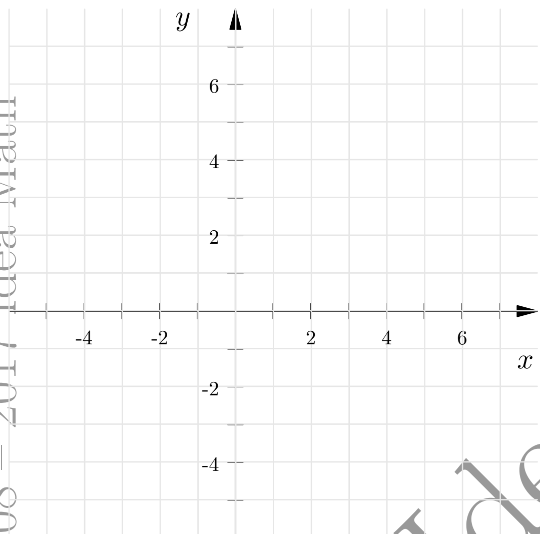
(a) $3x^2 - 9x$ (b) $20x^2 - 45$ (c) $x^2 + 5x - 7$ (d) $x^2 - 14x + 45$
(e) $x^2 + 3x - 18$ (f) $2x^2 + 3x - 7$ (g) $x^2 - x + 30$ (h) $3x^2 + 5x + 2$

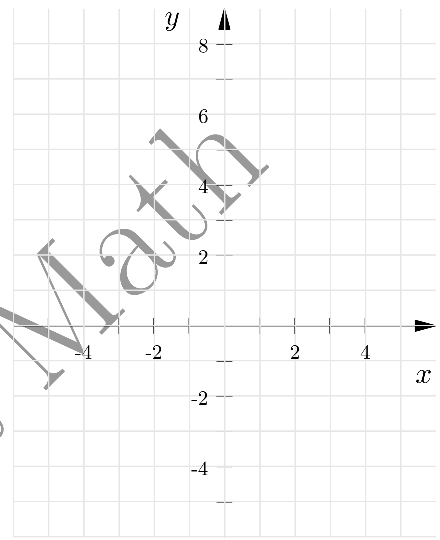
4. The product of two consecutive odd positive integers is equal to 4095. Set the numbers to be $2n-1$ and $2n+1$. Set up an equation for n and solve the equation to find the two numbers.

5. Factor $ax + bx - ay - by$.

4.2 Quadratic functions (part 2)

1. The graph of a quadratic function is called a *parabola*. This shape is common to all graphs of equations of the form $y = ax^2 + bx + c$, where a is nonzero. Use the coordinates system below on the left to plot the graphs of $y = x^2 - 1$ and $y = 4x - x^2$. How are these graphs alike, and how are they different?





2. What effect does the coefficient a have on the graph of the function $y = ax^2$? Use the coordinates system above on the right and sketch on the same axes the graphs of:

   (a) $y = 2x^2$ (b) $y = -\frac{1}{2}x^2$.

   How can you tell whether a is positive or negative by looking at the graph?

3. When the air traffic control radar range has doubled it has increased its area of surveillance by 10800 square miles. Find the new range of the radar.

4. A student swims upstream half a mile and then swims back to the starting point. The whole workout took two hours. If the speed of the student is 1 mile per hour, find the speed of of the current.

   Set the speed of the current to be x. Explain why x must satisfy the equation

$$\frac{0.5}{1+x} + \frac{0.5}{1-x} = 2.$$

   Simplify the equation and then solve it.

5. Find the x-intercepts of the following quadratic functions:

   (a) $y = x^2 - 4$ (b) $y = -3x^2 + 48$

   (c) $y = x^2 + 0.49$ (d) $y = 2x^2 - 7$

   Summarize by describing how to find the x-intercepts of any quadratic function $y = ax^2 + c$.

4.3 Distance, rate, and time

1. Sam and Cam have a lawn-mowing service. Their first job tomorrow morning is one that usually takes Sam 40 minutes to do alone, or Cam 30 minutes to do alone. This time they are going to team up, Sam starting at one side and Cam at the other side. The problem is to predict how many minutes it will take them to finish the job. What part of the lawn will Sam complete in the first ten minutes? What part of the lawn will Cam complete in the first ten minutes? What part of the lawn will the team complete in ten minutes? Set up a guess-and-check table with columns titled "minutes", "Sam part", "Cam part" and "Team part". What is the target value for the team part? Fill in two rows of the chart by making guesses in the minutes column. Then guess m and complete the solution algebraically.

2. In a 10-kilometer race, each runner runs 5 km to point P and returns to the start by the same route. Ian runs 4 kilometers per hour faster than Sean. Ian runs to point P, turns around, and meets Sean 4 km from the start. Assume that Ian and Sean each maintain a constant speed and start at the same time. What is Sean's time, in minutes, for running the 10 kilometers?

3. The hot-water faucet takes four minutes to fill the tub, and the cold-water faucet takes three minutes for the same job. How long to fill the tub if both faucets are used?

4. Bob ran half of the distance of the 24-kilometer race at 12 kilometers per hour, and he ran the other half at 8 kilometers per hour. Bill ran the same race, running half the time at 12 kilometers per hour and the other half at 8 kilometers per hour. Who wins the race? By how many minutes did the winner win the race?

5. It takes Franz 45 minutes to walk between her home and school. One morning she walked half way to school and remembered that she had left her calculator at home. She ran home and then ran all the way to school. It took 5 minutes at home to get her calculator. She runs twice as fast as she walks. How many minutes more than usual did it take her to get to school?

5.2 Sequences, series, growth, and decay (part 2)

1. To denote a sequence, we need index system. For example, we can use $(a_1, a_2, a_3, \ldots)$ to denote a sequence, where a_1 denotes the first term, a_2 denote the second term, and so on, a_n denotes the n^{th} term (a generic term) of the sequence. By the way, a_1, a_2, and a_n are usually read "a sub one", "a sub two", and "a sub n".

   Consider the *arithmetic* progression $(a_1, a_2, a_3, \ldots) = (19, 22, 25, \ldots)$.

   (a) What is a_{15}? What is a_{1728}?

   (b) Will 61 appear in this progression? Will 200 appear in this progression? If *yes*, which term is it? If *no*, explain the reason.

   (c) What is the n^{th} term of this sequence? (That is, express a_n in terms of n.)

2. A *geometric sequence* or *geometric progression* is a list of numbers in which each term is obtained by applying a constant multiplier to the preceding term.

   The first three terms of a *geometric* progression are 3, 12, 48.

   (a) Find the next five terms of the progression.

   (b) Will 6^6 appear in this sequence? Will $3 \cdot 2^{32}$ appear in this sequence? If *yes*, which term is it? If *no*, explain the reason.

   (c) What is the n^{th} term of this sequence?

   (d) Compute the product of the first ten terms of the progression.

3. Sometimes it is necessary to invest a certain amount of money at a fixed interest rate for a fixed number of years so that a financial goal is met. The parents of a child born today decide that \$55000 will be needed for the first year of the college expenses.

   When the child is born, they have deposited \$20000 in a money-market account that earns 6 percent annual interest. Suppose that no withdrawals or additional deposits are made.

   (a) Calculate how much money will be in the account one year later; two years later; three years later; t years later.

   (b) Will there be enough money in this account to cover the first year of the college expense when the child is 18 years old?

4. (Continuation) How does the exponential equation $y = 20000(1.06)^x$ relate to this problem. This is an example of *exponential growth*. Explain this term. Plot points (x, y) for $n = 0, 1, 2, 3, 4, 5$ in the coordinate plane.

   (a) Do these points lie on a straight line?

   (b) Sketch a curve passing through these points.

   (c) Use the graphing device to graph the curve $y = 20000(1.06)^x$ and compare this curve with the curve your drew in part (b).

(d) There is a point on the curve with x-coordinate equal to 0.5. Plot this point. Estimate the value of the y-coordinate of this point. What is the meaning of this value? Repeat this process for $x = 3.6$.

5. Suppose that a \$25,000 car loses 20% of its value every year.

   (a) How much is it worth after 1 year? 2 years? T years?

   (b) Write an exponential function to describe this decay.

   (c) Use a calculating device to graph your function in part (b). Make an accurate sketch of this graph on a piece of graph paper.

   (d) Write an equation whose solution represents the number of years it will take for the car to be worth half of its original value.

   (e) Use your sketch in part (c) to give an eyeball integer estimation to the equation in part (d). Use your calculator to confirm if your estimation is correct.

5.5 Solving quadratic equations (part 2)

1. When Ryan was asked to solve the equation $(t-3)(t-5)=0$, his answer was $t=3$ or $t=5$. How did Ryan obtain these answers? Was he correct? When Ryan was asked to solve the equation $(t-3)(t-5)=1$, his answer was $t=4$ or $t=6$. How did Ryan obtain these answers? Was he correct? Solve $(t+3)(t+5)=35$. We have discussed this property before. What is the name of this property?

2. When asked to solve the equation $x^2-3x=1$, Jamie thought, "I would like to add a number to both sides, but because I have 3, I can't find such integer to create a perfect square". Taylor said: "You are right, but that isn't a problem. You can add $\left(\frac{3}{2}\right)^2$ to both sides and still solve the equation.

   Complete the suggested approach and solve the equation.

3. Solving a quadratic equation by rewriting the left side as a perfect-square polynomial is called *solving by completing the square*. Use this method to solve each of the following equations. Leave your answers in exact form.

   (a) $x^2-8x-9=0$

   (b) $x^2+x-3=0$

4. (Continuation) By a similar *completing the square* process, one can convert $y=x^2-8x-9$ into the vertex form. Do so. Find the vertex and the axis of symmetry of the parabola $y=x^2-8x-9$.

5. Find the vertex and the axis of symmetry of the parabola $y=x^2+x-3$.